Model order reduction and composite control for a class of slow-fast systems around a non-hyperbolic point

H. Jardón-Kojakhmetov[1] and Jacquelien M.A. Scherpen[1]

Abstract—In this letter we investigate a class of slow-fast systems for which the classical model order reduction technique based on singular perturbations does not apply due to the lack of a Normally Hyperbolic critical manifold. We show, however, that there exists a class of slow-fast systems that after a well-defined change of coordinates have a Normally Hyperbolic critical manifold. This allows the use of model order reduction techniques and to qualitatively describe the dynamics from auxiliary reduced models even in the neighborhood of a non-hyperbolic point. As an important consequence of the model order reduction step, we show that it is possible to design composite controllers that stabilize the (non-hyperbolic) origin.

Index Terms—Model order reduction, Perturbation methods, Nonlinear control systems.

I. INTRODUCTION

MODEL order reduction is a technique often used to reduce the complexity of a system. There exist many model order reduction techniques [1], [2], and the particular type of method to be used usually depends on the structure and properties of the original system being studied.

For two-timescale systems, a classical model order reduction technique is based on singular perturbation methods. Basically, we decompose a two-timescale system into two lower dimensional ones [3]. One of them describes the dynamics in the slow timescale, while the other describes the behavior in the fast timescale. After the description of these two separate systems is performed, one is able to fully understand the dynamics of the original, two-timescale, plant. Moreover, thanks to the aforementioned decomposition, the design of controllers for two-timescale systems is greatly simplified. Applications of the latter type of model order reduction and the associated controller design are plenty, and can be found in robotics, communications, electronics, smart-grids, etc [3], [4], [5], [6]. However, model order reduction based on singular perturbations depends on a strong assumption, called Normal Hyperbolicity (see Section II for details). Whenever such condition is not fulfilled, then the (timescales) reduction method cannot be used.

Here we investigate a class of two-timescale systems for which, in principle, the model order reduction based on singular perturbations is not applicable. However, after a well-suited transformation, called *geometric desingularization*, we are able

[1]H. Jardón-Kojakhmetov and Jacquelien M.A. Scherpen are with the Jan C. Willems Center for Systems and Control and the Engineering and Technology Institute of the University of Groningen, The Netherlands {h.jardon.kojakhmetov,j.m.a.scherpen}@ rug.nl

to discover hidden hyperbolic slow-fast dynamics. This allows us to use classical model order reduction techniques (based on singular perturbations) to qualitatively describe the behavior of the original two-timescale system. Furthermore, thanks to the capability for model order reduction, we are also able to design controllers in the "composite-control" framework [3], see Section V.

II. PRELIMINARIES

We study slow-fast systems, which are of the form

$$\begin{aligned} \dot{x} &= f(x,z,\varepsilon) \\ \varepsilon\dot{z} &= g(x,z,\varepsilon), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^{n_s}$, $z \in \mathbb{R}^{n_f}$, $0 < \varepsilon \ll 1$, and $f(x,z,\varepsilon)$ and $g(x,z,\varepsilon)$ are sufficiently smooth functions. The independent time variable for (1) is t and therefore the over-dot stands for $\frac{d}{dt}$. When studying slow-fast systems one usually defines the fast-time variable $\tau = \frac{t}{\varepsilon}$, which allows to rewrite (1) as an ε-family of vector fields

$$X_\varepsilon : \begin{cases} x' &= \varepsilon f(x,z,\varepsilon) \\ z' &= g(x,z,\varepsilon), \end{cases} \tag{2}$$

where now the prime $'$ denotes derivative with respect to τ. Note that as long as $\varepsilon \neq 0$, the trajectories of (1) are equivalent to those of (2), the only difference is their time parametrization.

A. Model order reduction via singular perturbations

This method aims to take advantage of the structure of (1) in order to obtain two reduced models which, together, provide sufficiently good information of the trajectories of (1). Here we briefly describe the method. Let us start by taking the limit $\varepsilon \to 0$ of (1) and (2), thus defining the reduced systems

$$\begin{aligned} \dot{x} &= f(x,z,0) \\ 0 &= g(x,z,0), \end{aligned} \tag{3}$$

which is a Differential Algebraic Equation, and

$$X_0 : \begin{cases} x' &= 0 \\ z' &= g(x,z,0), \end{cases} \tag{4}$$

called the layer equation. The first ingredient is to define the critical manifold.

Definition 1 (Critical manifold). *The critical manifold S is the set of critical points of the vector field X_0, that is*

$$S = \left\{ (x,z) \in \mathbb{R}^{n_s + n_f} \,|\, g(x,z,0) = 0 \right\}. \tag{5}$$

Note that S is also the phase-space of the trajectories of the DAE (3). Next, an essential ingredient for model order reduction via singular perturbations is that the critical manifold S is Normally Hyperbolic.

Definition 2 (Normally Hyperbolic). *A point $s \in S$ is said to be hyperbolic if s is a hyperbolic equilibrium point of the reduced vector field $z' = g(x,z,0)$, where x is taken as a fixed parameter. The manifold S is called Normally Hyperbolic if every $s \in S$ is hyperbolic.*

Equivalently, a critical manifold is Normally Hyperbolic if and only if $\frac{\partial g}{\partial z}|_S$ has no eigenvalues with zero real part. If that is the case, by the implicit function theorem, there exists a unique local solution $z = Z(x)$ to the algebraic equation $g(x,z,0) = 0$ in a neighborhood of any $s \in S$. It follows that the DAE (3) is reduced to the *the slow subsystem*

$$\dot{x} = F(x), \tag{6}$$

where $F(x) = f(x,Z(x),0)$. The relationship between trajectories of the slow subsystem and those of the slow-fast system X_ε is explained by Fenichel's Theory [7], [8].

Theorem 1. *Consider a slow-fast system (1) or (2), and suppose that $S_0 \subset S$ is a compact normally hyperbolic submanifold. Then for $\varepsilon > 0$ sufficiently small, the following hold*

- *There exists a locally invariant manifold S_ε diffeomorphic to S_0.*
- *S_ε has distance of order $O(\varepsilon)$ (as $\varepsilon \to 0$) from S_0.*
- *The flow on S_ε converges to the flow on S_0 as $\varepsilon \to 0$.*
- *S_ε has the same stability properties as S_0.*
- *S_ε is usually not unique, but all such manifolds lie within distance $O(\exp(-c/\varepsilon))$ from each other. Any of such representatives is called the slow manifold.*

Simply put, Theorem 1 tells us that the reduced systems (6) and (4) provide a good enough approximation of the dynamics of a slow-fast system (1) or (2).

We must note that the timescale separation in (1) or (2) is explicit, defining the so-called "standard form". However, systems without explicit timescale separation (or in non-standard form) can also be treated [9], [10], [11]. Briefly speaking, a system $\dot{x} = F(x,\varepsilon)$ exhibiting slow and fast dynamics can be put into standard form (1) via a change of coordinates which may depend on ε. These systems, however, still need to satisfy certain hyperbolicity-type conditions in order to define the aforementioned change of coordinates.

In this article, we shall study a class of slow-fast systems, in standard form, that do not satisfy the Normal Hyperbolicity property. Thus, in principle, model order reduction as explained above is not applicable. However, the class of systems that we present below have a "hidden" Normally Hyperbolic critical manifold which is discovered after an appropriate change of coordinates.

In order to introduce the class of systems to be studied, we need the concept of quasihomogeneity.

Definition 3 (Quasihomogeneous function). *Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. The function f is said to be quasihomogeneous of quasidegree*

δ *and type $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ if and only if for all $\rho > 0$ holds*

$$f(\rho^{\alpha_1} x_1, \ldots, \rho^{\alpha_n} x_n) = \rho^\delta f(x_1, \ldots, x_n). \tag{7}$$

Note that if $\alpha = (1, \ldots, 1)$ we get the usual definition of a homogeneous function of degree δ. Below we shall deal with quasihomogeneous polynomials, for which the quasihomogeneity type and degree can be determined via the Newton Polygon method [12].

For simplicity, a quasihomogeneous object of type $\alpha = (\alpha_1, \ldots, \alpha_n)$ shall be called α-quasihomogeneous.

III. THE GEOMETRIC DESINGULARIZATION METHOD

The geometric desingularization method is a suitable technique employed to study the behavior of dynamical systems near a non-hyperbolic singularity. It was introduced in the context of slow-fast systems in [13]. Thanks to such technique many complex phenomena in the vicinity of a non-hyperbolic point have been elucidated. Here, we present a very brief description of the technique, for more details see e.g. [8], [14]. To better explain the method in the context of slow-fast systems, it is convenient to lift the family X_ε, given by (2), up and instead consider a single vector field on $\mathbb{R}^{n_s+n_f+1}$ of the form

$$X : \begin{cases} x' &= \varepsilon f(x,z,\varepsilon) \\ z' &= g(x,z,\varepsilon) \\ \varepsilon' &= 0. \end{cases} \tag{8}$$

For the rest of this article, we shall assume that the origin is a non-hyperbolic equilibrium point of X, that is $\frac{\partial g}{\partial z}(0,0,0)$ has at least one eigenvalue with zero real part. In an intuitive way, the geometric desingularization method transforms non-hyperbolic points of slow-fast systems to (partially) hyperbolic ones. The method consists of a well suited coordinate transformation called blow up. Such a transformation describes the system in generalized polar coordinates and is defined as follows.

Definition 4. *A (quasi-homogeneous)[1] blow up transformation is a map[2]*

$$\begin{aligned} \Phi \quad : \quad & \mathbb{S}^{n_s+n_f} \times \mathbb{R} \to \mathbb{R}^{n_s+n_f+1} \\ & \Phi(\bar{x}, \bar{z}, \bar{\varepsilon}, \rho) \mapsto (\rho^\alpha \bar{x}, \rho^\beta \bar{z}, \rho^\gamma \bar{\varepsilon}), \end{aligned} \tag{9}$$

where $(\bar{x}, \bar{z}, \bar{\varepsilon}) \in \mathbb{S}^{n_s+n_f}$ (where $\mathbb{S}^N$ denotes the N-sphere) $\rho \in [0,\infty)$, $\alpha \in \mathbb{N}^{n_s}$, $\beta \in \mathbb{N}^{n_f}$ and $\gamma \in \mathbb{N}$.

Note that the sphere $\mathbb{S}^{n_s+n_f} \times \{0\}$ is mapped to the origin of $\mathbb{R}^{n_s+n_f}$, which means that a neighborhood U of the origin can be related to a neighborhood V of the sphere $\mathbb{S}^{n_s+n_f} \times \{0\}$. The choice of the weights (α, β, γ) is, in principle arbitrary, but later will be related to the quasihomogeneity type of the vector field. The blow up map induces the "blown up" vector field $\hat{X}$ on $\mathbb{S}^{n_s+n_f} \times \mathbb{R}$, defined as $\hat{X} = D\Phi^{-1} \circ X \circ \Phi$, where $D\Phi$ denotes the differential of Φ. Usually, it happens that $\hat{X}$

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[1]A homogeneous blow up (or simply blow up) refers to all the exponents α, β, γ set to 1.

[2]Here, for simplicity of notation, $\rho^\alpha \bar{x} = (\rho^{\alpha_1} \bar{x}_1, \ldots, \rho^{\alpha_{n_s}} \bar{x}_{n_s})$, and similarly for $\rho^\beta \bar{z}$.

vanishes all along $\mathbb{S}^{n_s+n_f} \times \{0\}$. In that case one defines the *desingularized* vector field $\overline{X}$ by $\overline{X} = \frac{1}{\rho^m}\hat{X}$, where $m \in \mathbb{N}$ is as large as possible such that $\overline{X}$ is not degenerate along $\mathbb{S}^{n_s+n_f} \times \{0\}$. Then, note that $\hat{X}$ and $\overline{X}$ are smoothly equivalent for all $\rho > 0$. This means that the dynamics of $\overline{X}$ around the sphere $\mathbb{S}^{n_s+n_f} \times \{0\}$ are equivalent to those of $\hat{X}$ and in turn, the latter gives a qualitative description of the dynamics of X around the origin. The advantage of studying $\overline{X}$ is that, whenever the blow up map is well chosen, the singularities of $\overline{X}$ are (partially) hyperbolic, making its analysis much simpler than that of X.

The question on how to determine the appropriate weights of the blow up map for general systems of dimension greater than 2 is an open problem [8, Section 7.2 and 7.8]. However, as long as the singularity is not too degenerate, we can potentially desingularize a vector field via a finite number of blow ups. In this article we study a class of systems for which the blow up map is given by the properties of the class itself.

IV. MODEL ORDER REDUCTION VIA GEOMETRIC DESINGULARIZATION

Consider the slow-fast system (8) and let $\alpha = (\alpha_1, \ldots, \alpha_{n_s}) \in \mathbb{N}^{n_s}$, $\beta = (\beta_1, \ldots, \beta_{n_f}) \in \mathbb{R}^{n_f}$, $\gamma \in \mathbb{N}$, $q = (\alpha, \beta, \gamma) \in \mathbb{N}^{n_s+n_f+1}$ and $\delta \in \mathbb{N}$. We assume that the origin $(x,z) = (0,0) \in \mathbb{R}^{n_s} \times \mathbb{R}^{n_f}$ is an isolated equilibrium point of X for all $\varepsilon > 0$, and that X satisfies the following properties

P1. f_i is q-quasihomogeneous of quasidegree $\delta + \alpha_i$ for all $i = 1, \ldots, n_s$.
P2. g_i is q-quasihomogeneous of quasidegree $\delta + \beta_i$ for all $i = 1, \ldots, n_f$.
P3. $\frac{\partial g}{\partial z}(0,0,\varepsilon) \in O(\varepsilon)$. That is, in a neighborhood of the origin, the linear part of g vanishes as $\varepsilon \to 0$.

Remark 1. *For the class of systems studied here, the loss of normal hyperbolicity is due to P3 above. This is not the only way in which a slow-fast system may lose normal hyperbolicity. Another one is, e.g., due to degenerate singularities induced by nonlinear terms in the layer equation, compare with [14], [15], [16].*

Expanding $g(x,z,\varepsilon)$ around $z = 0$, and considering P3, we can write

$$g(x,z,\varepsilon) = A_0(x,\varepsilon) + \varepsilon A_1(x)z + O(z^2), \qquad (10)$$

where $A_0(0,\varepsilon) = 0$ and $A_1(x)$ is non-singular in a sufficiently large neighborhood of $x = 0$. Note that the Jacobian, at the origin, associated to the layer equation X_0 is nilpotent. Therefore, the classical model order reduction method via singular perturbations is not applicable. In order to be able to describe the behavior of X, we shall use the geometric desingularization technique. For this, let a blow up map (recall Definition 4) be defined by

$$x = \rho^\alpha \bar{x}, \ z = \rho^\beta \bar{z}, \ \varepsilon = \rho^\gamma \bar{\varepsilon}, \qquad (11)$$

where $(\bar{x}, \bar{z}, \bar{\varepsilon}) \in \mathbb{S}^{n_s+n_f}$, and $\rho \in [0, \infty)$.

Remark 2. *Note that we relate the weights of the blow up map to the quasihomogeneity type of the functions f and g.*

Remark 3. *Usually, it is difficult to study a vector field in spherical coordinates (the blown up vector field). To overcome such difficulty, one usually considers charts (parametrization of a hemisphere). The most important chart in the type of analysis presented here is the "central" chart defined as $K_{\bar{\varepsilon}} = \{\bar{\varepsilon} = 1\}$ [8], [13]. It is precisely in such a chart where the hidden hyperbolic slow-fast dynamics are found. Moreover, note that the chart $K_{\bar{\varepsilon}}$ corresponds to a small neighborhood of size $O(\varepsilon^{\alpha/\gamma}) \times O(\varepsilon^{\beta/\gamma})$ of the origin of (8).*

Proposition 1. *The desingularized vector field in the chart $K_{\bar{\varepsilon}} = \{\bar{\varepsilon} = 1\}$ reads as*

$$\begin{aligned} \rho' &= 0 \\ \bar{x}' &= \rho^\gamma \bar{f}(\bar{x}, \bar{z}) \\ \bar{z}' &= \bar{g}(\bar{x}, \bar{z}), \end{aligned} \qquad (12)$$

where $\bar{f}(\bar{x}, \bar{z}) = f(\bar{x}, \bar{z}, 1)$ and $\bar{g}(\bar{x}, \bar{z}) = g(\bar{x}, \bar{z}, 1)$.

Proof. The blow up map in chart $K_{\bar{\varepsilon}} = \{\bar{\varepsilon} = 1\}$ is given by

$$(x, z, \varepsilon) = (\rho^\alpha \bar{x}, \rho^\beta \bar{z}, \rho^\gamma), \qquad (13)$$

where, for simplicity, $\rho^\alpha \bar{x} = (\rho^{\alpha_1}\bar{x}_1, \ldots, \rho^{\alpha_{n_s}}\bar{x}_{n_s})$, and similarly for $\rho^\beta \bar{z}$. It follows from $\varepsilon' = 0$ in (8) and (13) that $\rho' = 0$. Next

$$\begin{aligned} \rho^{\alpha_i}\bar{x}_i' &= \rho^\gamma f_i(\rho^\alpha \bar{x}, \rho^\beta \bar{z}, \rho^\gamma) \\ &= \rho^{\gamma+\delta+\alpha_i} f_i(\bar{x}, \bar{z}, 1), \end{aligned} \qquad (14)$$

for all $i = 1, \ldots, n_s$. Using similar arguments for the rescaling of the z-variable we obtain

$$\begin{aligned} \rho' &= 0 \\ \bar{x}' &= \rho^{\gamma+\delta}\bar{f}(\bar{x}, \bar{z}) \\ \bar{z}' &= \rho^\delta \bar{g}(\bar{x}, \bar{z}). \end{aligned} \qquad (15)$$

Finally, the desingularized vector field is obtained by rescaling time by a factor $\frac{1}{\rho^\delta}$, leading to (12). Here, to avoid introducing more notation, we are recycling the prime $'$ to denote the derivative with respect to the aforementioned rescaled time. $\qquad \square$

Remark 4. *The loss of normal hyperbolicity in (8) is due to the fact that the linear part of g is of order $O(\varepsilon)$, see (10). However, note that with the change of coordinates of Proposition 1, the linear component of $\bar{g}$ becomes independent of the parameter.*

Next, we have the most important property of (12).

Theorem 2. *The desingularized vector field (12) with $0 < \rho \ll 1$ is a slow-fast system with Normally Hyperbolic critical manifold given by*

$$\bar{S} = \left\{ (\bar{x}, \bar{z}) \in \mathbb{R}^{n_s+n_f} \mid \bar{g}(\bar{x}, \bar{z}) = g(\bar{x}, \bar{z}, 1) = 0 \right\}. \qquad (16)$$

Proof. The proof follows from the fact that $A_1(\bar{x})$ (in (10)) is non-singular in a sufficiently large neighborhood of the origin $(\bar{x}, \bar{z}) = (0, 0)$. $\qquad \square$

At this point, in view of Theorem 2, all the classical theory of singular perturbations [3] apply to the blown up vector field (12). This leads to the main result of this document.

Theorem 3. *Under the assumptions of this section, up to time rescaling, and for $\varepsilon > 0$ sufficiently small, the solutions of (8) are locally equivalent to those of the reduced systems*

$$\dot{x} = f(\bar{x}, h(\bar{x}), 1), \qquad (17)$$

where $h(\bar{x})$ is the unique root of $g(\bar{x}, \bar{z}, 1) = 0$, and

$$\bar{z}' = g(\bar{x}, \bar{z}, 1). \qquad (18)$$

Proof. First, recall that X denotes the vector field (8), and Φ the blow up map (11). Let $\hat{X}$ denote the blown up vector field, that is $\hat{X} = D\Phi^{-1} \circ X \circ \Phi$, where $D\Phi$ denotes the Jacobian of Φ. Since Φ is a diffeomorphism on $\mathbb{S}^{n_s+n_f} \times (0, \infty)$, the vector fields X and $\hat{X}$ are conjugate for all $\rho > 0$. Next, the desingularized vector field $\overline{X}$ is defined by $\overline{X} = \frac{1}{\rho^\delta}\hat{X}$ (see details in Proposition 1). The vector fields $\hat{X}$ and $\overline{X}$ are smoothly equivalent for all $\rho > 0$. That is, their only difference is the timescale of their solutions.

According to Lemma 2, the vector filed $\overline{X}$ is slow-fast with a normally hyperbolic critical manifold. From the implicit function theorem, it follows that there exists a unique root $\bar{z} = h(\bar{x})$ of the algebraic equation $g(\bar{x}, \bar{z}, 1) = 0$. Then, the corresponding slow dynamics are given as (17). On the other hand from (12) it is evident that the "blown up" layer equation is (18), where $\bar{x}$ is treated as a fixed parameter. Next, by Geometric Singular Perturbation (Theorem 1 see also [7], [8]) it follows that within a compact neighborhood of the origin $(\bar{x}, \bar{z}) = (0,0)$, the solutions of the desingularized vector field $\overline{X}$ are of the form $\bar{x}(\tau_s) = \bar{x}_r(\tau_s) + O(\rho^\gamma)$ and $\bar{z}(\tau_f) = \bar{z}_r(\tau_f) + O(\rho^\gamma)$, where $\bar{x}_r$ and $\bar{z}_r$ are solutions of the slow and the layer equations respectively, and τ_s, τ_f denote the slow and fast times related to the slow-fast system $\overline{X}$. Finally we can go back to the qualitative behavior of the original vector field X by tracking back the transformations performed, first the desingularization between $\hat{X}$ and $\overline{X}$ and second the blow down (the inverse of the blow up map Φ) taking $\hat{X}$ to X. $\qquad \square$

Remark 5. *Hidden normally hyperbolic manifolds as described above frequently occur in the analysis of slow-fast systems, for example in the context of biochemical oscillations see [17], [18]. However, a general treatment of such scenario is not available. With this note we attempt to contribute to the description of a class of systems for which the existence of (hidden) slow-fast normally hyperbolic dynamics, after geometric desingularization, is expected. Such hidden Normally Hyperbolic properties are also of crucial importance for control purposes. See more details in Section V and in Example VI-B.*

V. COMPOSITE CONTROL

The theory presented above has important applications for controller design. In few words, in Section III we have shown that after a change of coordinates, the class of systems (8), satisfying properties P1-P3, have a Normally Hyperbolic structure. This implies that within the blow up space, the origin of the desingularized vector field can be stabilized via a composite controller in the spirit of [3]. The only requirement would

be that the controller is of the appropriate quasihomogeneity type to satisfy properties P1-P2, see an illustrative example in Section VI-B. Thus, it is natural that under the aforementioned conditions, the controller that stabilized the non-hyperbolic origin of (8) can also be decomposed. However, we currently study generalizations and cases where these conditions are not necessarily met. Related results shall be presented elsewhere by the authors.

VI. ILLUSTRATIVE ACADEMIC EXAMPLES

In this section we present a couple of examples showing the highlights of our contribution.

A. Example 1. (Model order reduction)

Consider the slow-fast system

$$\begin{aligned} x' &= -\varepsilon x^2 z \\ z' &= -\varepsilon z + x^2 \qquad (19) \\ \varepsilon' &= 0. \end{aligned}$$

Notice that the origin is an isolated equilibrium point, and that it is not possible to conclude anything about its stability via linearization. On the other hand, the corresponding DAE and layer equation are given by

$$\text{DAE:} \begin{cases} \dot{x} &= -x^2 z \\ 0 &= x \end{cases} \qquad \text{Layer:} \begin{cases} x' &= 0 \\ z' &= x^2 \end{cases} \qquad (20)$$

Thus, it is evident that it is quite difficult to conclude anything about the dynamics of (19) from the "reduced systems" (20). So we proceed by following Section III.

The polynomials $f(x, z, \varepsilon) = -x^2 z$ and $g(x, z, \varepsilon) = -\varepsilon z + x^2$ are quasihomogeneous of type $\alpha = (2, 1, 3)$ and quasidegree 5 and 4 respectively, that is, $\delta = 3$. According to the quasihomogeneity type α and quasidegree δ, the blow up map reads as

$$x = \rho^2 \bar{x}, \ \bar{z} = \rho \bar{z}, \ \varepsilon = \rho^3 \bar{\varepsilon}. \qquad (21)$$

The desingularized vector field[3] in the chart $K_{\bar{\varepsilon}}$ is given by

$$\begin{aligned} \rho' &= 0 \\ \bar{x}' &= -\rho^3 \bar{x}^2 \bar{z} \qquad (22) \\ \bar{z}' &= -\bar{z} + \bar{x}^2. \end{aligned}$$

Note that for $0 < \rho \ll 1$ (22) can be regarded as a slow-fast system with normally hyperbolic stable critical manifold $\bar{S} = \left\{ (\bar{x}, \bar{z}) \in \mathbb{R}^2 \mid \bar{z} = \bar{x}^2 \right\}$, and then the slow subsystem is

$$\dot{x} = -\bar{x}^4. \qquad (23)$$

It follows from a simple analysis that (23) has an isolated equilibrium point p at the origin, which is a (non-hyperbolic) saddle point; it is attracting (resp. repelling) for initial conditions $\bar{x}_0 > 0$ (resp. $\bar{x}_0 < 0$). On the other hand, the layer equation reads as

$$\bar{z}' = -\bar{z} + \bar{x}^2. \qquad (24)$$

[3]Recall that to obtain the desingularized vector field first one performs the blow up map, and then rescales by an appropriate monomial ρ^m.

One then concludes that every point $\bar{s} \in \bar{S}$ is a "global" exponentially stable equilibrium point of the layer equation (24) along each fiber $\bar{x} =$ constant.

From the previous analysis it follows that the dynamics of the reduced systems and of the slow-fast system (22) are as shown in Figure 1.
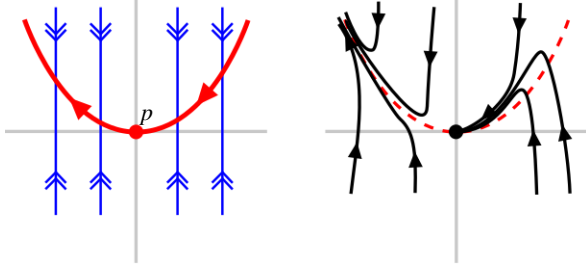

Fig. 1. The description of (22) is realized from the reduced subsystems (23) and (24), shown in the left side. The red thick curve depicts the slow manifold $\bar{S}$, and the dynamics within it are given by the slow flow (23). Transversally to $\bar{S}$, the dynamics are given by the layer equation (24), depicted in thin blue. The behavior for $\rho > 0$ is described by Fenichel's theory, and is shown in the right side.

We can now proceed with the *qualitative* description of the dynamics of (19) for $\varepsilon > 0$ sufficiently small. First, define the set $\Delta = \left\{ (x,z) \in \mathbb{R}^2 \,|\, \varepsilon z = x^2 \right\}$. Note that Δ and $\bar{S}$ are related via the blow up map (21), however, Δ is not a normally hyperbolic critical manifold according to Definition 2. Due to the qualitative analysis performed in the blown up space, we know that trajectories of (19) approach Δ up to a neighborhood U_ε of it. Within this neighborhood, the trajectories will be governed by the flow along Δ, which is equivalent to the flow along $\bar{S}$. The saddle point p is mapped to a saddle point $o = (0,0)$. Finally, we conclude that for $\varepsilon > 0$ sufficiently small, the flow of (19) is equivalent to that shown in Figure 1. This is evidenced in Figure 2, where we provide a simulation of the dynamics of (19).
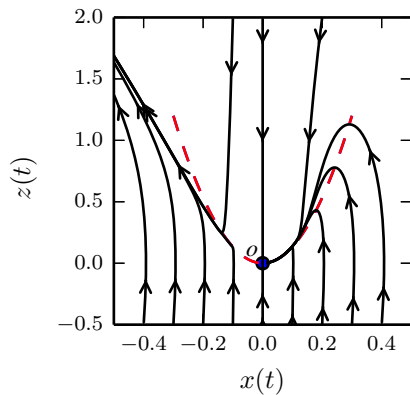

Fig. 2. Phase-portrait of (19) coinciding with the qualitative description provided via model order reduction. The dashed curve represents Δ and corresponds to a "hidden" slow manifold $\bar{S}$ found in the blown up space, compare with Figure 1. For simulation purposes we have used $\varepsilon = 0.05$.

B. Example 2. (Composite control)

Let us now illustrate how composite control can be used for the class of systems studied in this document. Consider the control system

$$
\begin{aligned}
x' &= -\varepsilon x^2 z \\
z' &= \varepsilon z + x^2 + u \\
\varepsilon' &= 0.
\end{aligned}
\tag{25}
$$

Following a similar analysis as in the previous example, the desingularized vector field reads as

$$
\begin{aligned}
\rho' &= 0 \\
\bar{x}' &= -\rho^3 \bar{x}^2 \bar{z} \\
\bar{z}' &= \bar{z} + \bar{x}^2 + \bar{u}(\bar{x}, \bar{z}),
\end{aligned}
\tag{26}
$$

where $\bar{u}(\bar{x}, \bar{z}) = u(\bar{x}, \bar{z}, 1)$ denotes the blow up of the controller u. The open-loop critical manifold is given by $\bar{S} = \left\{ \bar{z} = -\bar{x}^2 \right\}$. Moreover, $\bar{S}$ is unstable since the corresponding eigenvalue of the layer equation is $\lambda = 1$ for every point $\bar{s} \in \bar{S}$. The idea to stabilize the origin of (26) is to design the controller $\bar{u}$ using the composite methodology (see details in [3]). For this, one proposes $\bar{u}(\bar{x}, \bar{z}) = \bar{u}_s(\bar{x}) + \bar{u}_f(\bar{x}, \bar{z})$, where $\bar{u}_s$ denotes the slow controller, while $\bar{u}_f$ the fast one. A useful (but not necessary) property is that $\bar{u}_f$ vanishes along solutions of $0 = \bar{g}(\bar{x}, \bar{z}) + \bar{u}_s$. Thus, the corresponding DAE reads as

$$
\dot{\bar{x}} = -\bar{x}^2 z, \qquad 0 = \bar{z} + \bar{x}^2 + \bar{u}_s(\bar{x}),
\tag{27}
$$

where the algebraic equation has solution $\bar{z} = -(\bar{x}^2 + \bar{u}_s)$. It follows that the slow subsystem is simply given by $\dot{\bar{x}} = \bar{x}^2(\bar{x}^2 + \bar{u}_s)$. So, we can choose $\bar{u}_s(\bar{x}) = -\bar{x} - \bar{x}^2$, and then the closed-loop slow subsystem is given by $\dot{\bar{x}} = -\bar{x}^3$. It is straightforward to show[4] that $\bar{x} = 0$ is an asymptotically stable point of the closed-loop slow subsystem. Next, the layer equation reads as

$$
\bar{z}' = \bar{z} + \bar{x}^2 + \bar{u}_s(\bar{x}) + \bar{u}_f(\bar{x}, \bar{z}) = \bar{z} - \bar{x} + \bar{u}_f(\bar{x}, \bar{z}),
\tag{28}
$$

where $\bar{x}$ is treated as a fixed parameter. Next, let us propose $\bar{u}_f(\bar{x}, \bar{z}) = -2(\bar{z} - \bar{x})$, and therefore the closed-loop layer equation reads as $\bar{z}' = -(\bar{z} - \bar{x})$. It is again straightforward to show[5] that the set $\left\{ \bar{z} = \bar{x} \right\}$ is asymptotically stable for every $\bar{x} \in U_{\bar{x}}$, where $U_{\bar{x}} \in \mathbb{R}$ is any compact neighborhood of $\bar{x} = 0$. Then, the composite controller[6] is given by

$$
\bar{u} = \bar{x} - \bar{x}^2 - 2\bar{z}.
\tag{29}
$$

The complete closed-loop system (27) now reads as

$$
\begin{aligned}
\rho' &= 0 \\
\bar{x}' &= -\rho^3 \bar{x}^2 \bar{z} \\
\bar{z}' &= -\bar{z} + \bar{x},
\end{aligned}
\tag{30}
$$

where the corresponding closed-loop critical manifold is Normally Hyperbolic and stable, and the origin $(\bar{x}, \bar{z}) = (0,0)$ is asymptotically stable for $\rho > 0$ sufficiently small.

---

[4]Using a Lyapunov function $V(\bar{x}) = \frac{1}{4}\bar{x}^4$.

[5]Using a Lyapunov function $W(\bar{x}, \bar{z}) = \frac{1}{2}(\bar{z} - \bar{x})^2$.

[6]The proposed controller $\bar{u} = \bar{u}_s + \bar{u}_f$ also satisfies the so-called interconnection conditions of the composite control method (see Section 7.6 of [3]), but for simplicity of exposition, we have left those details out.

We now go back to the original coordinates to find the appropriate controller that stabilizes the origin of (25). The transformation performed to obtain the auxiliary slow-fast system was in two steps: first the blow up map, and then the time rescaling, see Section III. In this particular example, the aforementioned operations amount to the relation $u = \rho^4 \bar{u}$, that is

$$
\begin{aligned}
u(x,z,\varepsilon) = \rho^4 \bar{u}(\bar{x},\bar{z}) &= \rho^4 \bar{x} - \rho^4 \bar{x}^2 - 2\rho^4 \bar{z} \\
&= \varepsilon^{2/3} x - x^2 - 2\varepsilon z,
\end{aligned}
\tag{31}
$$

where the last equality is due to the blow up map (21). Note that, indeed, u given by (31) is a quasihomogeneous polynomial of type $\alpha = (2,1,3)$ and quasidegree 4, as required. A simulation of the corresponding trajectories and controller is shown in Figure 3.
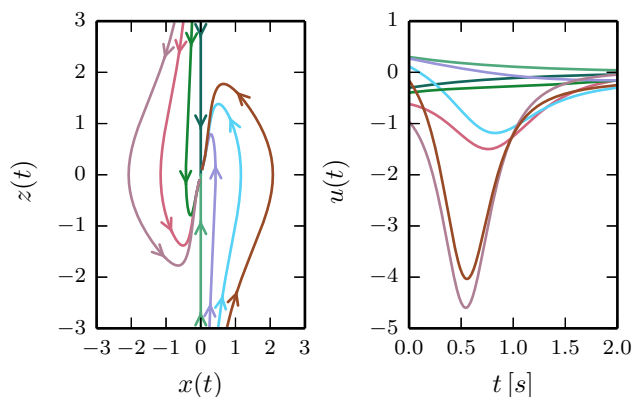

Fig. 3. Left: Phase-portrait of the closed-loop system (25) under the action of the controller (31). For the simulation we have used $\varepsilon = 0.05$. Right: Control signals for the different initial conditions of the phase-portrait.

VII. DISCUSSION AND CONCLUSIONS

In this document we have investigated a model reduction technique for a class of two-timescale systems which are not normally hyperbolic at the origin. In principle, the classical model order reduction technique based on singular perturbations is not applicable for such class of systems. However, we have shown that after an appropriate transformation, called geometric desingularization, the class of systems under study turns out to be suitable for decompositions into slow and fast components. Geometric desingularization and model order reduction based on singular perturbations have been combined successfully to describe the dynamics of a two-timescale system around a non-hyperbolic point. Moreover, we have shown, in an illustrative example, that thanks to the capability to use model order reduction in the blown up space, one is able to design controllers that stabilize a non-hyperbolic equilibrium point with the composite algorithm in the spirit of [3].

As future research, we plan to explore whether the model reduction method presented here can be extended to a bigger class of non-hyperbolic slow-fast systems. In particular, it would be interesting to apply our result to the reduction of large scale networks [19], [20] which may have degenerate dynamics. Moreover, the development of controllers for slow-fast systems with non-hyperbolic dynamics needs to be further investigated, particularly when more than two timescales are involved.

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