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Nonlinear adaptive stabilization of a class of planar slow-fast systems at a non-hyperbolic point

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***Abstract*— Non-hyperbolic points of slow-fast systems (also known as singularly perturbed ordinary differential equations) are responsible for many interesting behavior such as relaxation oscillations, canards, mixed-mode oscillations, etc. Recently, the authors have proposed a control strategy to stabilize non-hyperbolic points of planar slow-fast systems. Such strategy is based on geometric desingularization, which is a well suited technique to analyze the dynamics of slow-fast systems near non-hyperbolic points. This technique transforms the singular perturbation problem to an equivalent regular perturbation problem. This papers treats the nonlinear adaptive stabilization problem of slow-fast systems. The novelty is that the point to be stabilized is non-hyperbolic. The controller is designed by combining geometric desingularization and Lyapunov based techniques. Through the action of the controller, we basically inject a normally hyperbolic behavior to the fast variable. Our results are exemplified on the van der Pol oscillator.**

I. Introduction

This paper studies the nonlinear adaptive stabilization problem of a planar slow-fast system at a non-hyperbolic point. A slow-fast system (SFS) is a singularly perturbed ordinary differential equation which depends (in a singular way) on a small parameter ε. The presence of the small parameter ε is reflected in a number of variables evolving slowly while others evolve much faster. Such type of behavior can be found in, e.g., electric circuits, flexible-joint robot manipulators, chemical reaction models, ecological models, etc.

It is by now well known, e.g. [3], [7], [8], [11], [10], [9], that under hyperbolicity conditions the analysis of a SFS can be performed via two reduced-order systems. However, *many interesting phenomena can be attributed to the loss of hyperbolicity*. Qualitatively speaking, the trajectories of a SFS may jump when passing near a non-hyperbolic point. For example, relaxation oscillations [21] are found in slow-fast systems transitioning through a (non-hyperbolic) fold point.

Recently, see [5], the authors have proposed a control strategy, based on geometric desingularization, to stabilize planar slow-fast systems at a fold point. Briefly speaking, geometric desingularization (also known as blow up) is a change of coordinates and time rescaling that in some sense eliminates the singular dependence of the system on the small parameter ε [2], [14]. The ability to stabilize non-hyperbolic points becomes important when the operation point of a certain slow-fast system is to be set within a small neighborhood of a jump point, see Section V. In this paper we extend the the results of [5] to a more robust scenario: the adaptive stabilization of a non-hyperbolic point. The main idea is to combine the geometric desingularization method (Section II) with Lyapunov-based controller design. In this way, we are able to design a controller for slow-fast systems that stabilizes a non-hyperbolic operating point. Interestingly, the controller actually injects a hyperbolic behavior to the fast variable.

The rest of this document is arranged as follows: in Section II we provide the essential background information leading to our results. Afterwards, in Section III, we formally state our main contribution. Next, we prove our main result by designing a nonlinear adaptive controller based on the geometric desingularization and Lyapunov methods. Then, in Section V we exemplify the implications of our proposed method by implementing an adaptive controller on the van der Pol oscillator. We shall show that it is possible to stabilize an operating point defined at a jump point. Finally, in Section VI we provide some concluding remarks and digress on future research.

II. Preliminaries

A planar SFS is a singularly perturbed ordinary differential equation of the form

$$\begin{aligned} \dot{x} &= f(x,z,\varepsilon) \\ \varepsilon\dot{z} &= g(x,z,\varepsilon), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}$, $z \in \mathbb{R}$, and $0 < \varepsilon \ll 1$ is a small parameter. We assume that the functions $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^3 \to \mathbb{R}$ are C^∞, and uniformly bounded in ε. Thus, the presence of the small parameter ε implies that the variable z evolves much faster than x. In this sense, x and z are called the slow and fast variables respectively. When $\varepsilon > 0$, a new time parameter τ is defined by the relation $t = \varepsilon\tau$. Then (1) can be rewritten as an ε-parameter family of vector fields

$$X_\varepsilon : \begin{cases} x' &= \varepsilon f(x,z,\varepsilon) \\ z' &= g(x,z,\varepsilon), \end{cases} \tag{2}$$

where now the prime denotes the derivative with respect to τ. Note that whenever $\varepsilon > 0$ and $g \not\equiv 0$, the systems (1) and (2) are equivalent, their only difference is the timescale. Then, it is completely equivalent to study (1) or (2).

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A common strategy to study the dynamics of SFSs is to consider first the limit equations when $\varepsilon \to 0$. Accordingly, in this limit (1) becomes a DAE (Differential Algrebraic Equation) of the form

$$\begin{aligned} \dot{x} &= f(x,z,0) \\ 0 &= g(x,z,0), \end{aligned} \tag{3}$$

while (2) becomes a layer equation which reads as

$$X_0 : \begin{cases} x' &= 0 \\ z' &= g(x,z,0), \end{cases} \tag{4}$$

where x has the rôle of a fixed parameter. Related to these two limit equations, the following important set is defined.

Definition 1: The 1-dimensional *critical manifold* S is defined as

$$S = \left\{(x,z) \in \mathbb{R}^2 \,|\, g(x,z,0) = 0\right\}. \tag{5}$$

Remark 1: The critical manifold S serves as the phase space of the DAE (3) and as the set of equilibrium points of the layer equation (4).

A subset $S_0 \subset S$ is called normally hyperbolic if each point $p \in S_0$ is a hyperbolic equilibrium point of (4). In the context of this document, normal hyperbolicity is equivalently given as $\partial_z g(p) \neq 0$[1] for each point $p \in S_0$. It follows from the implicit function theorem that S_0 can be locally expressed by a graph $z = h_0(x)$, where h_0 is a smooth function. Therefore (3) can be rewritten as

$$\dot{x} = f(x, h_0(x), 0), \tag{6}$$

which is called *the reduced slow-subsystem*. We remark that the flow of (6) is the same as that of (3) along the critical manifold S_0. The main results of Geometric Singular Perturbation Theory (GSPT), see e.g. [3], [7], [8], show that for $\varepsilon > 0$ but sufficiently small (and under hyperbolicity conditions), the critical manifold S_0 persists as an invariant manifold S_ε of the SFS (2). Therefore, the flow of (6) provides a good approximation, of order $O(\varepsilon)$, of the flow of (2) along S_ε. Moreover, the associated stable and unstable manifolds of S_0 are also perturbed to stable and unstable manifolds of S_ε. All this means that, near hyperbolic points of S_0, the dynamics of (2) can be fully understood by studying the reduced systems (3) and (4). Applications in control systems, under the previously described hyperbolicity property and the implied reducibility, are many, see e.g. [9], [17], [20].

On the other hand, SFSs with a critical manifold having non-hyperbolic points have been used to model jumps in e.g. electric circuits, chemical reactions, ecological models, etc. From the mathematical point of view, the description of the flow near non-hyperbolic points becomes much more complicated than in the hyperbolic scenario. In this paper, we are interested in designing an adaptive controller for planar SFSs at points where the hyperbolicity condition is lost. For this, we shall use a combination of the geometric desingularization technique and Lyapunov based control methods. Let us now proceed by briefly describing the geometric desingularization technique.

[1]Along this document ∂_z shall denote the partial derivative with respect to z.

A. Geometric desingularization

The geometric desingularization technique, also known as blow up, is a quite successful geometric mathematical method allowing the analysis of SFSs near non-hyperbolic points [2], [14], [15]. To start, let us expand (2) by adding the trivial equation $\varepsilon' = 0$. In this way, we do not deal with a family of vector fields anymore but with a three dimensional vector field given by

$$X : \begin{cases} x' &= \varepsilon f(x,z,\varepsilon) \\ z' &= g(x,z,\varepsilon) \\ \varepsilon' &= 0. \end{cases} \tag{7}$$

Briefly speaking, geometric desingularization is a well suited local change of coordinates under which a non-hyperbolic singularity, say at the origin, of (7) is simplified. By this we mean that after the coordinate transformation, the new singularities of the induced vector field are hyperbolic or semi-hyperbolic. Such a change of coordinates is of the form

$$x = r^{\alpha_1}\bar{x},\ z = r^{\alpha_2}\bar{z},\ \varepsilon = r^{\alpha_3}\bar{\varepsilon}, \tag{8}$$

where $(\bar{x},\bar{z},\bar{\varepsilon}) \in \mathbb{S}^2$ and $r \in [0,\infty)$, and where $\alpha_1, \alpha_2, \alpha_3$ are suitable positive integers. Since we have assumed that $\varepsilon > 0$, we may also assume that $\bar{\varepsilon} \in [0,\infty)$. Let $\Phi : \mathbb{S}^2 \times [0,\infty) \to \mathbb{R}^3$ denote the blow up map (8). Note that Φ maps the sphere $\mathbb{S}^2 \times \{0\}$ to the origin of $\mathbb{R}^3$. Moreover, the map Φ induces a vector field $\tilde{X}$ defined by $\Phi_* \tilde{X} = X$. It may happen that $\tilde{X}$ is degenerate along $\mathbb{S}^2 \times \{0\}$ in which case one defines a new vector field $\bar{X}$ by $\bar{X} = \frac{1}{r^m}\tilde{X}$ for a suitable integer m such that $\bar{X}$ is not degenerate at $\mathbb{S}^2 \times \{0\}$. In this way, the dynamics of $\tilde{X}$ and $\bar{X}$ are equivalent outside $\mathbb{S}^2 \times \{0\}$ and thus it is equally useful to study $\bar{X}$. Then, one obtains a complete description of the dynamics of X around the origin by studying $\bar{X}$ around $\mathbb{S}^2 \times [0, r_0)$ for $r_0 > 0$.

When studying SFSs of dimension greater than 2 it is more convenient to use *charts* [2], [15]. A chart is a parametrization of a certain hemisphere of $\mathbb{S}^2 \times [0, r_0)$. More precisely in our particular problem, the charts are defined by

$$\begin{aligned} &K_{\pm\bar{x}} = \{\bar{x} = \pm 1\},\ K_{\pm\bar{z}} = \{\bar{z} = \pm 1\}, \\ &K_{\bar{\varepsilon}} = \{\bar{\varepsilon} = 1\}. \end{aligned} \tag{9}$$

Thus, the strategy is to study the flow of the induced vector field in the charts and later glue them together in order to describe the flow of X near the origin. We will show in the following sections that a controller designed for the blown up vector field $\bar{X}$ induces a controller for X. Moreover, the closed-loop behavior of $\bar{X}$ is carried over to X. An important observation is the following.

Lemma 1: Let N be an n-dimensional smooth manifold and $Y : N \to TN$, where TN denotes the tangent bundle of N, a smooth vector field. Let $\Psi : N \to M$ be a smooth local diffeomorphism (e.g a blow up map). Denote by Z the induced vector field defined by $Z\circ\Psi = D\Phi\circ Y$. Suppose that Σ_Y is a set of locally stable hyperbolic equilibrium points of Y. Then the set $\Psi(\Sigma_Y)$ is a set of locally stable hyperbolic equilibrium points of Z.

Proof: Just recall that Lyapunov functions are invariant under change of coordinates, see also [5]. ■

III. STATEMENT OF THE MAIN RESULT

For simplicity of exposition, let us consider the following SF control system

$$X: \begin{cases} x' &= \varepsilon(a_0 + a_1 x + a_2 z + u) \\ z' &= -(z^2 + x) \\ \varepsilon' &= 0, \end{cases} \tag{10}$$

where $a_i \in \mathbb{R}$ for $i = 0, 1, 2$ are assumed to be *unknown*. Let $a = (a_0, a_1, a_2)$ and denote by $\hat{a}$ the estimation of a. The goal is to design an adaptive controller $u = u(x, z, \varepsilon, \hat{a})$, with update law $\hat{a}' = h(x, z, \hat{a})$ in such a way that, for $\varepsilon > 0$ but sufficiently small the origin of X becomes locally asymptotically stable.

Remark 2:

- We opt to study (10) because it is one of the simplest non-linear SFSs having a non-hyperbolic singularity at the origin.[2] However, it shall be clear from our exposition that the results presented below can be easily extended to SFSs with one fast variable and m slow variables of the form
$$\begin{aligned} x' &= \varepsilon(A_0 + L(x, z, \varepsilon) + Bu) \\ z' &= -(z^2 + x_1), \end{aligned} \tag{11}$$
where $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$, $z \in \mathbb{R}$, $A_0 \in \mathbb{R}^m$, $B \in \mathbb{R}^m$ and $L : \mathbb{R}^{m+2} \to \mathbb{R}^m$ is a linear map.
- The negative sign of the z' equation is just for convenience of exposition. The same results shown below (up to minor and straightforward modifications) hold for the case $z' = z^2 + x_1$.
- We propose to treat the under-actuated system (10) instead of the fully actuated situation. Such a scenario is relevant for systems where only the slow variables can actually be modified by some external intervention (a control input), see for example [22]. To overcome this, we shall show that it is possible to "inject" a normally hyperbolic behavior to the fast variable through the backstepping algorithm [12].
- A general treatment of a more general class of SFSs around a fold point shall be presented in [6].

Our main contribution is as follows.

Theorem 1: Consider the slow-fast control system (10). Assume that $\text{sign}(a_0 - \hat{a}_0)$ is known. Then, if $\varepsilon > 0$ but sufficiently small, the adaptive controller $u = u(x, z, \hat{a}_0)$ given by

$$u = \frac{1}{\varepsilon}\left(-\varepsilon\hat{a}_0 + (2z - c_0\varepsilon^{1/3})(z^2 + x) + \varepsilon^{2/3}z - c_1\varepsilon^{1/3}y\right) \tag{12}$$

with update law

$$\hat{a}_0' = \gamma_0(\varepsilon^{-2/3}y + d_0\text{sign}(a_0 - \hat{a}_0)), \tag{13}$$

where $y = x + z^2 - c_0\varepsilon^{1/3}z$, with $c_0 > 0$, $c_1 > 0$, and $d_0 > 0$ renders the origin $(x, z) = (0, 0)$ as a locally asymptotically stable equilibrium point of (10).

Proof: The proof is given in Section IV-B and follows from the exposition of Section IV-A. ■

Remark 3: Observe that in Theorem 1, we do not require the estimation of the parameters a_1 and a_2. Moreover, we insist that knowing the sign of the error $a_0 - \hat{a}_0$ does not necessarily imply the knowledge of a_0. That is, in fact, a common assumption in adaptive control design, see [12], [13].

IV. NONLINEAR ADAPTIVE CONTROLLER DESIGN VIA GEOMETRIC DESINGULARIZATION

We shall first design the controller u of (10) in the blown up space $S^2 \times [0, \infty)$. In this document, we will only consider the chart $K_{\bar{\varepsilon}}$ (see (9)).

Remark 4: The analysis of the other charts is non-trivial and for some goals may be necessary. However, including the analysis in the other charts would make this document inappropriately long. On the technical side, the chart $K_{\bar{\varepsilon}}$ is the must relevant for the analysis performed in this paper. This is because, in such chart, we desingularize the effects of the singular parameter ε. The dynamics in the chart $K_{\bar{\varepsilon}}$ are equivalent to the dynamics of the original SF control system in a small neighborhood, of size $O(\varepsilon^{2/3}) \times O(\varepsilon^{1/3})$, of the origin (see (14)).

A. Analysis in the chart $K_{\bar{\varepsilon}}$

In this chart, the blow up map reads as

$$x = r^2\bar{x},\ z = r\bar{z},\ \varepsilon = r^3, \tag{14}$$

where the exponents are chosen so that the induced vector field has a common factor r. The induced vector field is obtained, as described in Section II, by:

1) Performing the blow up transformation, and
2) Divide by a factor r to get a non-degenerate vector field along $\{r = 0\}$.

After the aforementioned transformation, the SFS (10) reads as

$$\bar{X}: \begin{cases} r' &= 0 \\ \bar{x}' &= a_0 + a_1 r^2\bar{x} + a_2 r\bar{z} + \bar{u} \\ \bar{z}' &= -(\bar{z}^2 + \bar{x}), \end{cases} \tag{15}$$

where the prime denotes the derivative with respect to the rescaled time. Note that in this chart, $r \geq 0$ is a fixed parameter. Note also that the singular dependence on the parameter ε has disappeared and we can treat $\bar{X}$ as just a nonlinear control system. Moreover (15) is a regular perturbation problem in r. This means that it shall be sufficient to study the vector field $\bar{X}$ restricted to $r = 0$. Then, by regular perturbation arguments, the flow of $\bar{X}$ with $r > 0$ but sufficiently small is equivalent to the flow of $\bar{X}|_{r=0}$. So,

[2]A more mathematical reason is that fold singularities are generic in one parameter families of smooth maps [1], in this case $g(x, z) = z^2 + x$.

the task is to design a nonlinear adaptive controller $\bar{u}$ with an appropriate update law so that the origin is an asymptotically stable equilibrium point of $\bar{X}$.

Proposition 1: Consider the vector field $\bar{X}$. Assume $\text{sign}(a_0 - \hat{a}_0)$ is known. Then, for $r \geq 0$ but sufficiently small, the adaptive nonlinear controller

$$\bar{u} = -\hat{a}_0 + (2\bar{z} - c_0)(\bar{z}^2 + \bar{x}) + \bar{z} - c_1\bar{y}, \tag{16}$$

where $c_0 > 0$, $c_1 > 0$, $\bar{y} = \bar{x} + \bar{z}^2 - c_0\bar{z}$, and with update law

$$\hat{a}_0' = \gamma_0\left(\bar{y} + d_0\text{sign}(a_0 - \hat{a}_0)\right) \tag{17}$$

where $d_0 > 0$, $\gamma_0 > 0$, renders the origin as an asymptotically stable equilibrium point of $\bar{X}$.

Proof: The proof is based on the backstepping algorithm [12] and regular perturbation arguments [16]. Therefore, along this proof we restrict the vector field $\bar{X}$ to the subset $\{r = 0\}$. First, consider the "fast"-subsystem $\bar{z}' = -\bar{z}^2 - \bar{x}$ and treat $\bar{x}$ as a virtual controller. Let $V_0(\bar{z}) = \frac{1}{2}\bar{z}^2$ be a candidate Lyapunov function for such a subsystem. Then it is straightforward to show that the virtual controller $\bar{x} = \alpha_0(\bar{z}) = -\bar{z}^2 + c_0\bar{z}$, $c_0 > 0$, renders the point $\bar{z} = 0$ as a globally asymptotically stable equilibrium point of the subsystem $\bar{z}' = -\bar{z}^2 - \bar{x}$.

Remark 5: Under the choice of $\alpha_0(\bar{z}) = -\bar{z}^2 + c_0\bar{z}$, the "fast"-subsystem in (15) becomes $\bar{z}' = -c_0\bar{z}$. Qualitatively speaking, when $\bar{x} \approx \alpha_0$ the "fast" dynamics of (15) become normally hyperbolic. Due to the equivalence between (15) and (10), such hyperbolicity property is carried over (10) via blow down[3] (for $\varepsilon > 0$ but sufficiently small).

Next, consider the system (15), restricted to $\{r = 0\}$, written in coordinates $(\bar{y}, \bar{z})$, that is

$$\begin{aligned} \bar{y}' &= a_0 + (2\bar{z} - c_0)(-\bar{y} - c_0\bar{z}) + \bar{u} \\ \bar{z}' &= -(\bar{y} + c_0\bar{z}). \end{aligned} \tag{18}$$

Let $V_1(\bar{y}, \bar{z}) = V_0(\bar{z}) + \frac{1}{2}\bar{y}^2$ be a candidate Lyapunov function. It follows that

$$\begin{aligned} V_1' &= V_0' + \bar{y}\bar{y}' = \bar{z}\bar{z}' + \bar{y}\bar{y}' \\ &= -c_0\bar{z}^2 + \bar{y}(a_0 - (2\bar{z} - c_0)(\bar{y} + c_0\bar{z}) - \bar{z} + \bar{u}), \end{aligned} \tag{19}$$

Inspecting (19), we choose the controller $\bar{u}$ as

$$\bar{u} = -a_0 + (2\bar{z} - c_0)(\bar{y} + c_0\bar{z}) + \bar{z} - c_1\bar{y}, \tag{20}$$

with $c_1 > 0$. Then $V_1' = -c_0\bar{z}^2 - c_1\bar{y}^2$. Therefore, the origin $(\bar{y}, \bar{z}) = (0, 0)$ is an asymptotically stable equilibrium point of (18).

Finally, to design the adaptive update law, let us consider the extended system

$$\begin{aligned} \bar{x}' &= a_0 + \bar{u} \\ \bar{z}' &= -(\bar{z}^2 + \bar{x}) \\ \hat{a}_0' &= \gamma_0\bar{h}_0, \end{aligned} \tag{21}$$

where $\bar{h}_0$ is going to be chosen below. Let $\tilde{a}_0 = a_0 - \hat{a}_0$, then $\tilde{a}_0' = -\hat{a}_0'$. Propose the Lyapunov function

$$V(\bar{y}, \bar{z}, \tilde{a}) = V_1 + \frac{1}{2\gamma_0}\tilde{a}_0^2. \tag{22}$$

So, using (21) we have $V' = V_1' - \tilde{a}_0\bar{h}_0$. Substituting the controller

$$\bar{u} = -\hat{a}_0 + (2\bar{z} - c_0)(\bar{y} + c_0\bar{z}) + \bar{z} - c_1\bar{y}, \tag{23}$$

into V' we get

$$V' = -c_0\bar{z}^2 - c_1\bar{y}^2 + \tilde{a}_0(\bar{y} - \bar{h}_0). \tag{24}$$

Assuming that $\text{sign}(\tilde{a})$ is known (recall Remark 3), we can propose an update law of the form

$$\bar{h}_0 = \bar{y} + d_0\text{sign}(\tilde{a}_0), \tag{25}$$

where $d_0 > 0$. With such an update law V' reads as

$$V' = -c_0\bar{z}^2 - c_1\bar{y}^2 - d_0\tilde{a}_0\text{sign}(\tilde{a}_0). \tag{26}$$

Since $\tilde{a}_0\text{sign}(\tilde{a}_0) = |\tilde{a}_0|$ is a positive definite function of $\tilde{a}_0$, we conclude that V' is negative definite. Therefore we have that $(\bar{y}, \bar{z}, \tilde{a}) = 0$ is an asymptotically stable equilibrium point of (21). Finally, it is straightforward to check that $(\bar{y}, \bar{z}) \to (0, 0)$ implies $(\bar{x}, \bar{z}) \to (0, 0)$ as $t \to \infty$. Up to here we have proved that the controller (23) stabilizes the origin of $\bar{X}|_{r=0}$. The proof of the proposition is concluded by noting that the origin of $\bar{X}|_{r=0}$ is locally asymptotically stable and by regular perturbation arguments [16]. ■

In Figure 1 we show some closed-loop trajectories and their corresponding parameter estimation error.

B. *Proof of Theorem 1*

In the previous section, we have designed a nonlinear adaptive controller for the blown up vector field $\bar{X}$ given by (15). Due to Lemma 1, we can now blow down the controller and its update law of Proposition 1 so that the stability properties of $\bar{X}$ are carried over to the SFS X given by (10). Thus, according to the blow up map (14) and the result of Proposition 1 (in particular (23)), we can now compute the controller u and its update law in (x, z, ε) coordinates. Carrying out these computation one obtains

$$\begin{aligned} u &= \frac{1}{\varepsilon}\left(-\varepsilon\hat{a}_0 + (2z - c_0\varepsilon^{1/3})(z^2 + x) + \varepsilon^{2/3}z - c_1\varepsilon^{1/3}y\right) \\ \hat{a}_0' &= \gamma_0(\varepsilon^{-2/3}y + d_0\text{sign}(\tilde{a}_0)), \end{aligned} \tag{27}$$

where $y = x + z^2 - c_0\varepsilon^{1/3}z$. We note that the positive constants c_0, c_1, d_0, and γ_0 serve as tunning parameters of the controller. This concludes the proof of Theorem 1.

Remark 6: From (27), we see that the control u is not smooth in ε. This is a non-trivial fact and is related to the non-hyperbolicity of the fold point (the origin), this is well justified in e.g. [2], [14], [15]. Note, however that $\lim_{(x,z,\varepsilon)\to(0,0,0)} u = -\hat{a}_0$.

In Figure 2 we show the corresponding blown down trajectories (in (x, z) coordinates) of the those of Figure 1.

[3]Blow down stands for the inverse of the blow up map.

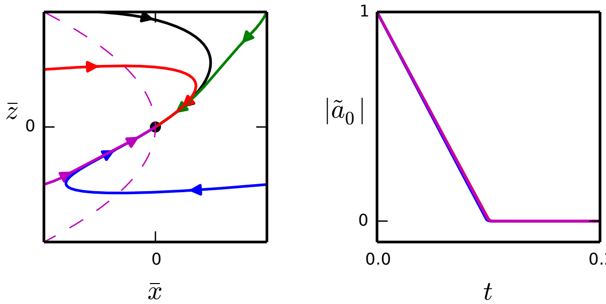

Fig. 1. Left: trajectories of the closed-loop system (15) under the nonlinear adaptive controller of Proposition 1. Right: the error norm $|\tilde{a}_0| = |a_0 - \hat{a}_0|$ of the trajectories plotted on the left. For these plots we have used parameters $(a_0, a_1, a_2, r, c_0, c_1, d_0, \gamma_0) = (1, 2, 3, 0.5, 1, 7, 100, 0.1)$, and initial condition $\hat{a}_0(0) = 0$.

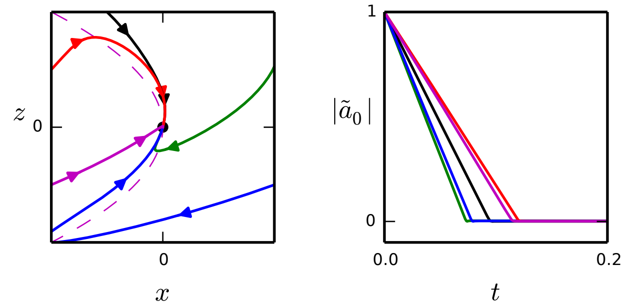

Fig. 2. Left: trajectories of the blown down closed-loop system (10) under the nonlinear adaptive controller of Theorem 1. Right: the error norm $|\tilde{a}_0| = |a_0 - \hat{a}_0|$ of the trajectories plotted on the left. For these plots we have used parameters $(a_0, a_1, a_2, \varepsilon, c_0, c_1, d_0, \gamma_0) = (1, 2, 3, 0.01, 1, 7, 100, 0.1)$, and initial condition $\hat{a}_0(0) = 0$. Compare with Fig. 1.

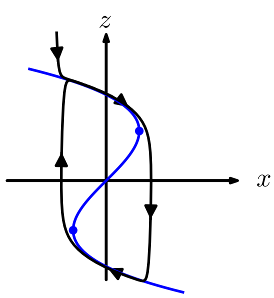

Fig. 3. Open-loop phase portrait of the van der Pol oscillator (28). The (blue) S-shaped manifold represents the critical manifold S defined by (29). The system has two fold (jump) points located at $\pm\left(\frac{2}{3}, 1\right)$.

V. Example: The van der Pol oscillator

To exemplify the adaptive controller designed in this document, let us study a relaxation oscillator in the form of a van der Pol Oscillator. Thus, consider the SF control system

$$\begin{aligned} \dot{x} &= az + u \\ \varepsilon\dot{z} &= -\left(\frac{1}{3}z^3 - z + x\right), \end{aligned} \tag{28}$$

where $a \neq 0$ is an unknown scalar and $u = u(x, z, \varepsilon)$ is the control signal. The system defined by (28) has an isolated, unstable, equilibrium point at the origin and a unique, asymptotically stable limit cycle. The corresponding critical manifold is given by

$$S = \left\{(x, z) \in \mathbb{R}^2 \,|\, \frac{1}{3}z^3 - z + x = 0\right\} \tag{29}$$

The phase portrait of the open-loop system is shown in Figure 3.

The goal is to stabilize one of the fold (non-hyperbolic) points, say $p = \left(\frac{2}{3}, 1\right)$. To this end, first we move the origin to such a point. This is done by the translation $\tilde{x} = x - \frac{2}{3}$, $\tilde{z} = z - 1$. Thus we obtain the system

$$\begin{aligned} \dot{\tilde{x}} &= a + a\tilde{z} + \tilde{u} \\ \varepsilon\dot{\tilde{z}} &= -\left(\tilde{z}^2 + \tilde{x} + \frac{1}{3}\tilde{z}^3\right), \end{aligned} \tag{30}$$

where $\tilde{u}$ stands for the controller in the new coordinates.

Remark 7: Up to leading order terms (30) fits in the context presented in the main part of this document. This is also visible from the shape of S in Figure 3.

By following Theorem 1, we know that in order to stabilize the origin of (30), we can choose the controller $\tilde{u}$ as

$$\tilde{u} = \frac{1}{\varepsilon}\left(-\varepsilon\hat{a} + (2\tilde{z} - \varepsilon^{1/3})(\tilde{z}^2 + \tilde{x}) + \varepsilon^{2/3}\tilde{z} - \varepsilon^{1/3}\tilde{y}\right) \tag{31}$$

with update law

$$\hat{a}' = \gamma(\varepsilon^{-2/3}\tilde{y} + d\mathrm{sign}(a - \hat{a})), \tag{32}$$

where $\tilde{y} = \tilde{x} + \tilde{z}^2 - c_0\varepsilon^{1/3}\tilde{z}$, and all constants d, c_0, c_1, γ are positive. By returning to the original coordinates (x, z) we obtain the controller u for (28) of the form

$$\begin{aligned} u =& \frac{1}{\varepsilon}\left(-\varepsilon\hat{a} + \left(2z + 2 - \varepsilon^{1/3}\right)\left(z^2 - 2z + x + \frac{1}{3}\right) + \right. \\ & \left. \varepsilon^{2/3}(z-1) - \varepsilon^{1/3}y\right) \\ \hat{a}' &= \gamma\varepsilon^{-2/3}\left(y + d\mathrm{sign}(a - \hat{a})\right) \\ y &= z^2 - 2z + x + \frac{1}{3} - c_0\varepsilon^{1/3}(z-1). \end{aligned} \tag{33}$$

The controller (33) is then applied to the van der Pol system (28). The simulation results are shown in Figure 4, where we have used the parameters $(a, c_0, c_1, d, \gamma, \varepsilon) = (1, 1, 1, 100, .1, .05)$ and the initial conditions $(x(0), z(0), \hat{a}(0)) = (-1, 3, 0)$.

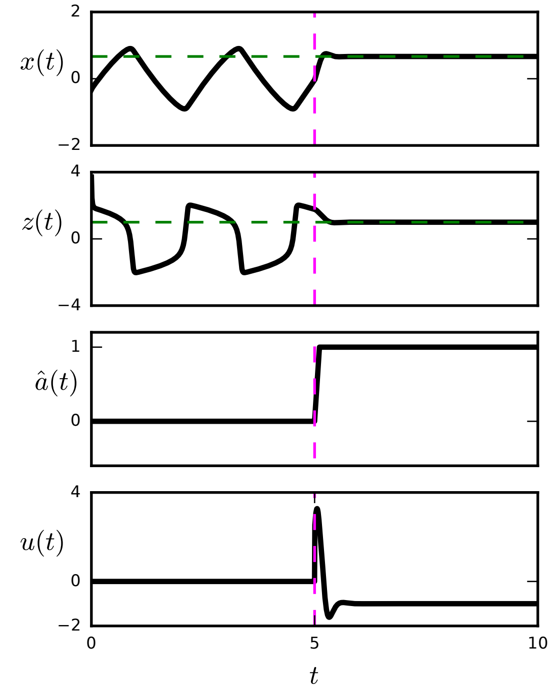

Fig. 4. Closed-loop signals of the van der Pol oscillator (28). In the first 5 seconds, the system is left in open-loop. At $t = 5$ the controller (33) is applied. From the graphics it is possible to see that the convergence to the desired operating point $p = \left(\frac{2}{3}, 1\right)$ is quick. Also, the estimation of the unknown parameter $a = 1$ is done almost instantaneously. Finally, note that the control input u is bounded. In fact, as noted in Remark 6, u takes the limit value of $u = -\hat{a} = -a = -1$.

VI. Conclusions and perspectives

In this paper we study the problem of nonlinear adaptive control to stabilize a non-hyperbolic point. We propose to use a combination of the geometric desingularization technique with Lyapunov controller design strategy. By doing so, we are able to study a regular perturbation problem, in the blown up space, instead of a singular one. In this space, we propose a nonlinear adaptive controller for which the Lyapunov functions are constructed via the backstepping method. In this way, we are able to inject a normally hyperbolic behavior to the fast variable and stabilize a non-hyperbolic point.

From the results presented in this document, several questions arise. First, a generalization of the SFS studied here is given by

$$\begin{aligned} x' &= \varepsilon\left(f(x,z,\varepsilon) + u(x,z,\varepsilon)\right) \\ z' &= -\left(z^k + \sum_{i=1}^{k-1} x_i z^{i-1} + O(\varepsilon)\right), \end{aligned} \tag{34}$$

where $x \in \mathbb{R}^{k-1}$, $z \in \mathbb{R}$. Observe that in (34), the origin is a highly degenerate singularity. Moreover, there are nonempty s-dimensional, $s = 1, 2, \ldots, k-2$, subsets of non-hyperbolic singularities, which complicate the analysis. Systems of the form (34) may be suitable to model complicated phenomena in biology, chemistry, ecology, etc. Regarding the controller design, it is interesting to investigate if the non-smoothness w.r.t. ε of the controller design in this paper can be avoided. Another extension of the current work is to study the trajectory tracking problem, in particular along sets of non-hyperbolicity. For example, make a SFS follow a reference along a line of folds. Finally, another complicated problem is to avoid injecting a normally hyperbolic behavior to the fast variable and instead design controllers such that the trajectories of the closed loop system follow the slow manifold. This requires a thorough analysis of suitable Lyapunov functions for e.g., systems like (34).

References

[1] V.I. Arnold, S.M. Gusein-Zade, and A.N. Varchenko. *Singularities of Differentiable Maps, Volume I*, volume 17. Birkhäuser, 1985.

[2] Freddy Dumortier and Robert H. Roussarie. *Canard Cycles and Center Manifolds*, volume 121. American Mathematical Society, 1996.

[3] N. Fenichel. Geometric singular perturbation theory. *JDE*, pages 53–98, 1979.

[4] H. Jardón-Kojakhmetov, Henk W. Broer, and R. Roussarie. Analysis of a slow-fast system near a cusp singularity. *Journal of Differential Equations*, volume 260, issue 4, pages 3785–3843, 2016.

[5] H. Jardón-Kojakhmetov and Jacquelien M.A. Scherpen. Stabilization of a planar slow-fast system at a non-hyperbolic point. *Proceedings of the 22nd International Symposium on Mathematical Theory of Networks and Systems*, 2016.

[6] H. Jardón-Kojakhmetov, Jacquelien M.A. Scherpen and D. del Puerto-Flores. Stabilization of slow-fast systems at fold points. *Submitted*, 2017.

[7] C.K.R.T Jones. Geometric singular perturbation theory. In *Dynamical Systems*, LNM 1609, pages 44–120. Springer-Verlag, 1995.

[8] T.J. Kaper. An Introduction to Geometric Methods and Dynamical Systems Theory for Singular Peturbation Problems. In *Symposia in Applied Mathematics*, volume 56, pages 85–131. AMS, 1999.

[9] P.. Kokotović. Applications of Singular Perturbation Techniques to Control Problems. *SIAM Review*, 26(4), pages 501–550, 1984.

[10] P. Kokotović, John O'Reilly, and Hassan K. Khalil. *Singular Perturbation Methods in Control: Analysis and Design*. Academic Press, Inc., Orlando, FL, USA, 1986.

[11] P. Kokotović, R.E. O'Malley, and P. Sannuti. Singular perturbations and order reduction in control theory - An overview. *Automatica*, 12(2), pages 123–132, 1976.

[12] M. Krstić and P. Kokotović. Control Lyapunov functions for adaptive nonlinear stabilisation. *Systems and Control Letters*, 26, pages 17–23, 1995.

[13] M. Krstić, I. Kanellakopoulos and P. Kokotović. Nonlinear and adaptive control design Wiley, 1995.

[14] M. Krupa and P. Szmolyan. Extending geometric singular perturbation theory to non hyperbolic points: fold and canard points in two dimensions. *SIAM J. Math. Anal.*, 33, pages 286–314, 2001.

[15] M. Krupa and P. Szmolyan. Geometric analysis of the singularly perturbed planar fold. In *Multiple-Time.Scale Dynamical Systems*, LNM 1609, pages 89–116. Springer-Verlag, 2001.

[16] J. Murdock. *Perturbations: Theory and Methods*. Wiley-Interscience, 1991.

[17] M.C. Readman and P.R. Belanger. Stabilization of the Fast Modes of a Flexible-Joint Robot. *The International Journal of Robotics Research*, 11(2), pages 123–134, apr 1992.

[18] G. Reissig. Differential-algebraic equations and impasse points *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 43 (2), pages 122–133, 1996.

[19] S. Sastry and C. Desoer. Jump behavior of circuits and systems. *IEEE Transactions on Circuits and Systems*, 28(12), pages 1109–1124, dec 1981.

[20] M. W. Spong. Modeling and Control of Elastic Joint Robots. *Journal of Dynamic Systems, Measurement, and Control*, 109(4), 1987.

[21] B. van der Pol and J. van der Mark. The heartbeat considered as a relaxation oscillation, and an electrical model of the heart. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, Ser.7,6, pages 763–775, 1928.

[22] E. C. Zeeman. Differential equations for the heart beat and nerve impulse. In Towards a theoretical biology (Vol. 4, pp. 8-67). Edinburgh University Press.