Stabilization of a class of slow-fast control systems at non-hyperbolic points

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Abstract

In this document, we deal with the local asymptotic stabilization problem of a class of slow-fast systems (or singularly perturbed Ordinary Differential Equations). The class of systems studied here has the following properties: 1) they have one fast and an arbitrary number of slow variables, and 2) they have a non-hyperbolic singularity at the origin of arbitrary degeneracy. Our goal is to stabilize such a point. The presence of the aforementioned singularity complicates the analysis and the controller design. In particular, the classical theory of singular perturbations cannot be used. We propose a novel design process based on *geometric desingularization*, which allows the stabilization of a non-hyperbolic point of singularly perturbed control systems. Our results are exemplified on a didactic example and on an electric circuit.

Key words: Nonlinear control; Slow-fast systems; singular perturbations.

1 Introduction

Slow-fast systems are characterized by having more than one timescale. There are many phenomena in nature that behave in two or more timescales such as population dynamics, cell division, electrical circuits, power networks, chemical reactions, neuronal activity, etc. [20, 33, 24]. A particular property of slow-fast systems (under certain hyperbolicity conditions) is that they have a structure suitable for model order reduction. Simply put, certain slow-fast systems can be decomposed into two simpler subsystems, the slow and the fast. The analysis of those two subsystems allows a complete understanding of the more complex and higher dimensional one. A mathematical theory supporting the previous fact is Geometric Singular Perturbation Theory (GSPT) [7], see also [19] in the context of control systems. The above, however, relies on the strong assumption of global timescale separation. When that does not hold, the classical technique of model order reduction cannot be used. Thus, new mathematical tools need to be introduced in order to deal with problems without global timescale separation.

Regarding the latter situation, many interesting phenomena are characterized by not having a global timescale separation. This means that the variables of the system do not always have the same timescale relation throughout the phase-space. Mathematically speaking, this phenomenon is characterized by singularities of the critical manifold, see Section 2; and in a qualitative sense, one usually observes jumps in the phase portrait of the slow-fast system. Such an effect is also called *loss of normal hyperbolicity*. Prototypical examples of two-timescale systems without global timescale separation are the van der Pol oscillator [38], neuronal models [33], and electrical circuits with impasse points [3, 4, 28]. Since loss of normal hyperbolicity is present in many models, there is an increased need of their accurate understanding. The distinction between the global and the non-global timescale sepa-

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[1] H. Jardón-Kojakhmetov is supported by the Technical University Foundation Fellowship (TUFF) of the Technical University of Munich.

[2] D. del Puerto Flores is supported in part by the internal project PROSNI-2018 and the Mexican PRODEP project UDG-PTC-1319.

ration is much more than qualitative. From an analysis point of view, the available techniques are quite different; while the hyperbolic case is well established, the non-hyperbolic scenario still presents many challenges.

In the context of control systems, slow-fast systems with global timescale separation are nowadays well understood and have been used in many applications, e.g. [19, 30, 35]. The main and powerful idea in the classical context is to design controllers for the reduced subsystems, which later guarantee stability of the overall slow-fast system. On the other hand, slow-fast systems with non-hyperbolic points are far from being well understood, and to the best knowledge of the authors, there are still many open problems (see [8, 9, 18] for some particular results). From a dynamical systems perspective, the technique called *geometric desingularization* [6, 21] has been used to understand the complex behavior of slow-fast systems around non-hyperbolic points. In this regard, the authors have made preliminary progress in bringing such technique to the control systems community [15, 17] in the planar case.

The main contribution of this document is the development of a control design method based on geometric desingularization. This provides a solution to the problem of the stabilization of singularly perturbed control problems at a non-hyperbolic point. Remarkably, we further consider control systems which are only actuated on the slow variables. Although the technique presented here is completely different from the classical one [19], the idea remains the same: to obtain simpler subsystems where the control design becomes more accessible.

The rest of this document is organized as follows: in Section 2 we provide preliminary information to place our research into context. In Section 3 our main contribution is formally stated. Next, in Section 4 we briefly describe the geometric desingularization method and then we apply it to control systems in Section 5. Afterwards, in Section 6, we develop a controller based on the method previously introduced. Interestingly, we show that it is possible to inject a hyperbolic behavior to a slow-fast control system near a non-hyperbolic point even though the fast variable is not actuated. Later, in Section 7 we exemplify our results with a couple of numerical simulations. We finish in Section 8 with some concluding remarks and a digression on open problems.

2 Preliminaries

Abbreviations: SFS stands for Slow-Fast System, SFCS for Slow-Fast Control System, and NH for Normally Hyperbolic.

Notation: $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ denote the fields of real, integer, and natural numbers respectively. Given a field $\mathbb{F}$, $\mathbb{F}^n$ denotes the n-cross-product $\mathbb{F}\times\cdots\times\mathbb{F}$. The dimension of the slow and fast variables is n_s and n_f respectively, and $N = n_s + n_f$. Given $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$, we denote $x^\alpha = (x_1^{\alpha_1}, \ldots, x_n^{\alpha_n})$. The symbol $\mathbb{S}^n$ denotes the n-sphere. The parameter ε is always assumed $0 < \varepsilon \ll 1$. Let $\mathcal{N}$ be an n-dimensional manifold, a vector field $\mathcal{X} : \mathcal{N} \to T\mathcal{N}$ is written as $\mathcal{X} = \sum_{i=1}^n \mathcal{X}_i \frac{\partial}{\partial x_i}$, where $(x_1, \ldots, x_n)$ is a coordinate system in $\mathcal{N}$.

A slow-fast system (SFS) is a singularly perturbed ordinary differential equation of the form

$$\begin{aligned} \dot{x} &= f(x, z, \varepsilon) \\ \varepsilon\dot{z} &= g(x, z, \varepsilon), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^{n_s}$ (slow variable), $z \in \mathbb{R}^{n_f}$ (fast variable), f and g are sufficiently smooth functions, and the independent variable is the slow time t. One can also define a new time parameter $\tau = \frac{t}{\varepsilon}$ called the fast time, and then (1) is rewritten as

$$\begin{aligned} x' &= \varepsilon f(x, z, \varepsilon) \\ z' &= g(x, z, \varepsilon), \end{aligned} \tag{2}$$

where the prime $'$ denotes derivative with respect to τ. Note that (1) and (2) are equivalent as long as $\varepsilon > 0$. For convenience of notation sometimes we refer to (2) as $\mathcal{X}_\varepsilon$, and write (2) as the ε-family of vector fields $\mathcal{X}_\varepsilon = \varepsilon f(x, z, \varepsilon)\frac{\partial}{\partial x} + g(x, z, \varepsilon)\frac{\partial}{\partial z}$.

A first step towards understanding the dynamics of (1) or (2) is to consider the limit equations when $\varepsilon \to 0$. When we set $\varepsilon = 0$ in (1)-(2) we obtain the so-called Differential Algebraic Equation (DAE) and Layer Equation respectively, and are given by

$$\text{DAE:} \begin{cases} \dot{x} &= f(x, z, 0) \\ 0 &= g(x, z, 0) \end{cases} \qquad \text{Layer:} \begin{cases} x' &= 0 \\ z' &= g(x, z, 0) \end{cases}$$

These two reduced systems are not equivalent anymore, however, the critical manifold draws a bridge between them.

Definition 1 *The* critical manifold *of a slow-fast system is defined as*

$$\mathcal{S} = \left\{(x, z) \in \mathbb{R}^{n_s} \times \mathbb{R}^{n_f} \mid g(x, z, 0) = 0\right\}.$$

Note that $\mathcal{S}$ is n_s-dimensional and serves as the phase-space of the DAE and as the set of equilibrium points of the Layer equation. An important property of critical manifolds is *normal hyperbolicity.*

Definition 2 *A point $s \in \mathcal{S}$ is called hyperbolic if it is a hyperbolic equilibrium point of the reduced vector field $z' = g(x, z, 0)$ of the Layer equation. The manifold $\mathcal{S}$ is* *called normally hyperbolic (NH) if each point $s \in \mathcal{S}$ is hyperbolic.*

The importance of NH critical manifolds is explained by Geometric Singular Perturbation Theory (GSPT) [7, 24], which we summarize in the following theorem.

Theorem 3 (Fenichel [7, 24]) *Let $\mathcal{S}_0 \subseteq \mathcal{S}$ be a compact and NH invariant manifold of a SFS. Then, for $\varepsilon > 0$ sufficiently small, the following hold*

- *There exists a locally invariant manifold $\mathcal{S}_\varepsilon$ which is diffeomorphic to $\mathcal{S}_0$ and lies within distance of order $O(\varepsilon)$ from $\mathcal{S}_0$.*
- *The flow of the SFS X_ε along $\mathcal{S}_\varepsilon$ converges to the flow of the DAE along $\mathcal{S}_0$ as $\varepsilon \to 0$.*
- *$\mathcal{S}_\varepsilon$ has the same stability properties as $\mathcal{S}_0$.*

In simple terms Fenichel's theory guarantees that the flow of a slow-fast system (1) can be regarded as a small perturbation of that of the corresponding DAE and Layer equation.

In contrast, here we consider SFSs for which there exists a point $p \in \mathcal{S}$ such that the matrix $\frac{\partial g}{\partial z}(p)$ has at least one eigenvalue with zero real part. The loss of normal hyperbolicity can be related to jumps or rapid transitions in, e.g., biological systems, climate models, chemical reactions, nonlinear electric circuits, or neuron models [20, 22, 38, 5, 29]. Before presenting our contribution, let us recall the idea of composite control.

2.1 Composite control of slow-fast control systems

Let us define a slow-fast control system (SFCS) as

$$\begin{aligned} \dot{x} &= f(x, z, \varepsilon, u) \\ \varepsilon \dot{z} &= g(x, z, \varepsilon, u), \end{aligned} \tag{3}$$

where $x \in \mathbb{R}^{n_s}$, $z \in \mathbb{R}^{n_f}$, and $u \in \mathbb{R}^m$ is a control input. Let us now briefly recall a classical method to design controllers for SFCS with a NH critical manifold, for the rigorous exposition refer to [19]. Assume that the critical manifold of (3) is NH in a compact set $(x, z, u) \in U_x \times U_z \times U_u \subset \mathbb{R}^{n_s} \times \mathbb{R}^{n_f} \times \mathbb{R}^m$. The goal is to design a control u that stabilizes the origin $(x, z) = (0, 0) \in \mathbb{R}^{n_s} \times \mathbb{R}^{n_f}$ for $\varepsilon > 0$ sufficiently small. The hyperbolicity assumption implies that the critical manifold $\mathcal{S} = \{g(x, z, u) = 0\}$ can be locally expressed as a graph $z = h(x, u)$ for $(x, u) \in U_x \times U_u$. The idea of composite control is to design u as a sum of two simpler controllers, namely $u = u_s + u_f$ where $u_s = u_s(x)$ is "the slow controller" and $u_f = u_f(x, z)$ is "the fast controller". When designing u, u_f must be chosen so that it does not destroy the normal hyperbolicity of the system, meaning that $g(x, z, u_s(x) + u_f(x, z)) = 0$ must have (locally) a unique root $z = H(x)$. Thus, it is often required that the effects of the fast controller u_f disappear along the $\mathcal{S}$, that is $u_f|_\mathcal{S} = 0$. In this way $z = h(x, u_s(x))$ is (locally) a unique root of $g(x, z, u) = 0$ and the reduced flow along $\mathcal{S}$ is given by

$$\dot{x} = f(x, H(x), u_s(x)). \tag{4}$$

Note that the reduced system is, as expected, independent of the fast variable z and of the fast controller u_f. Thus, the slow controller u_s is designed to make $\{x = 0\}$ an asymptotically equilibrium point of (4). After u_s has been designed, one studies the layer problem $z' = g(x, z, u_s(x) + u_f(x, z))$, where x is taken as a fixed parameter, and where the fast controller is designed so that $z = h(x, u_s(x))$ is a set of asymptotically stable equilibrium points. The previous strategy plus some extra (technical) interconnection conditions guarantee that the origin $(x, z) = (0, 0) \in \mathbb{R}^{n_s} \times \mathbb{R}^{n_f}$ is an asymptotically stable equilibrium point of the closed-loop system (3) for $\varepsilon > 0$ but sufficiently small [19, Section 7]. This feature of normal hyperbolicity has been exploited in many applications, e.g. [10, 25, 31, 27, 39, 37].

3 Statement of problem and of the main result

It is evident that the composite control strategy described in Section 2.1 is not applicable around non-hyperbolic points.

In this article, we consider SFCSs with one fast variable ($n_f = 1$) and an arbitrary number of slow variables (n_s). To start, we consider that the critical manifold has a particular geometric structure.

Definition 4 (Non-hyperbolic point) *Consider the SFS (3) with $u = 0$ and let $p = (x_0, z_0) \in \mathcal{S}$. If there exists an integer $k \geq 2$ such that $g(x_0, z_0, 0) = \frac{\partial g}{\partial z}(x_0, z_0, 0) = \cdots = \frac{\partial^{k-1} g}{\partial z^{k-1}}(x_0, z_0, 0) = 0$ and $\frac{\partial^k g}{\partial z^k}(x_0, z_0, 0) \neq 0$ then p is called a non-hyperbolic point.*

We now make four defining assumptions on (3).

A1. The SFCS (3) has a non-hyperbolic point at the origin.
A2. The SFCS (3) is affine in the control.
A3. The SFCS (3) is slowly actuated. That is, there is no controller action on the fast variable z, and $u \in \mathbb{R}^{n_s}$. The fully actuated case is non-trivial but simpler; for example, we can first force the origin to be a hyperbolic equilibrium point of the closed loop reduced system $z' = g(x, z, \varepsilon, u)$ and then use the composite control idea described in Section 2.1.
A4. $f(x, z, \varepsilon, 0) = f_0(x, z) + \varepsilon f_1(x, z, \varepsilon)$ with $f_1(0, 0, \varepsilon) = 0$. This ensures that ε does not introduce any constant drift on the open-loop system near the origin.

Proposition 5 *Under the above assumptions the slow-fast control system* (3) *can be written as*

$$\begin{aligned} \dot{x} &= f(x,z,\varepsilon) + B(x,z,\varepsilon)u(x,z,\varepsilon) \\ \varepsilon\dot{z} &= -\left(z^k + \sum_{i=1}^{k-1} x_i z^{i-1}\right) + H(x,z,\varepsilon), \end{aligned} \tag{5}$$

where B is invertible and $H(x,z,\varepsilon)$ denotes higher order terms[3] .

PROOF. The form of the equation for $\dot{x}$ follows directly from the assumptions. Now we focus on the form of $g(x,z,0)$ in (3). From singularity theory, in particular Malgrange's preparation theorem [11], it is well-known that under A1., the function $g(x,z,0)$ is locally equivalent to

$$g(x,z,0) = \pm\left(z^k + \sum_{i=1}^{k} a_i(x) z^{i-1}\right) + O(z^{k+1}),$$

where the functions $a_i(x)$ are smooth and $a_i(0) = 0$. First, a linear change of coordinates $z \mapsto \alpha(x)z + \beta(x)$ for some well defined anc computable functions α and β allows us the get rid of the term z^{k-1}. Next, we can expand each $a_i(x)$ and assume that $\frac{\partial a_i}{\partial x_i}(0) \neq 0$. After all, although in singularity theory $(x_1,\ldots,x_k)$ play the role of smooth parameters, here we identify them with slow variables. Finally we group the higher order terms and write them as in (5). The choice of the negative sign in (5) is non-essential and minor modifications of the method presented here follow otherwise. □

Notation: To simplify our exposition let us denote from now on $G_k(x,z) = z^k + \sum_{i=1}^{k-1} x_i z^{i-1}$.

We now present our main result.

Theorem 6 *Consider the SFCS* (5). *Let us denote the i-th component of the vector Bu as $(Bu)_i$. Suppose the controller u is designed such as*

$$\begin{aligned} (Bu)_1 &= -A_1 + \varepsilon^{\frac{-1}{2k-1}}(1+c_0c_1)z + \varepsilon^{\frac{-k}{2k-1}}\sum_{i=2}^{k-1} c_i x_i z^{i-1} \\ &\quad + \varepsilon^{-1}\left(\frac{\partial G_k}{\partial z} - \varepsilon^{\frac{k-1}{2k-1}}(c_0+c_1)\right) G_k \\ (Bu)_i &= -A_i - c_i \varepsilon^{\frac{-k}{2k-1}} x_i, \\ (Bu)_j &= -A_j - c_j \varepsilon^{\frac{-k}{2k-1}} x_j, \end{aligned} \tag{6}$$

where all constants c_0, c_1, c_i, c_j are positive with $c_i \ll c_1$ for $i = 0, 2, \ldots, k-1$, $j = k, \ldots, n_s$. Then the origin $(x,z) = (0,0) \in \mathbb{R}^{n_s} \times \mathbb{R}$ is rendered locally asymptotically stable for $\varepsilon > 0$ sufficiently small.

The proof of Theorem 6 is given in Section 6.2 and is obtained by following the forthcoming sections.

Remark 7 *In Theorem 6 we only require the knowledge of $A = f(0,0,0)$, and can be extended to an adaptive version where A may be assumed unknown, see [17].*

4 The Geometric Desingularization method

Here we present a brief description of the Geometric Desingularization method. For more details see [24]. First of all, note that (2) is an ε-parameter family of N-dimensional[4] vector fields. For their analysis it is more convenient to lift such family up and consider instead a single $(N+1)$-dimensional vector field defined as

$$\mathcal{X}: \begin{cases} x' &= \varepsilon f(x,z,\varepsilon) \\ z' &= g(x,z,\varepsilon) \\ \varepsilon' &= 0 \end{cases} \tag{7}$$

The *geometric desingularization* method, also known as blow up, is a geometric tool introduced in [6] for the analysis of SFSs around non-hyperbolic points, see also [14, 20, 22, 23, 24, 36]. In an intuitive way, the blow up method transforms non-hyperbolic points of SFSs to (partially) hyperbolic ones.

Definition 8 *Consider a generalized polar coordinate transformation*

$$\begin{aligned} &\Phi: \mathbb{S}^N \times I \to \mathbb{R}^{N+1} \\ &\quad \Phi(\bar{x},\bar{z},\bar{\varepsilon},\bar{r}) \mapsto (\bar{r}^\alpha \bar{x}, \bar{r}^\beta \bar{z}, \bar{r}^\gamma \bar{\varepsilon}) = (x,z,\varepsilon), \end{aligned} \tag{8}$$

where $\sum_{i=1}^{n_s} \bar{x}_i^2 + \sum_{j=1}^{n_f} \bar{z}_j^2 + \bar{\varepsilon}^2 = 1$ and $\bar{r} \in I$ where I is a (possibly infinite) interval containing $0 \in \mathbb{R}$. A (quasi-homogeneous)[5] blow up *is defined by $(\bar{x},\bar{z},\bar{\varepsilon},\bar{r}) = \Phi^{-1}(x,z,\varepsilon)$. The inverse of the blow up is called* blow down[6] .

In many applications, and in particular in this paper, it is enough to consider $\bar{r} \in [0, r_0)$, $0 < r_0 < \infty$, which implies $\bar{\varepsilon} > 0$. Thus, let $\mathcal{B} = \mathbb{S}^N \times [0, r_0)$, and $\mathcal{Z} = \mathbb{S}^N \times \{0\}$. We now define the blow up the vector field $\mathcal{X}$.

[3] The choice of the negative sign in front of the fast equation is just for convenience, and a similar analysis as the one performed here follows otherwise.

[4] Recall that $N = n_s + n_f$.

[5] A homogeneous blow up (or simply blow up) refers to all the exponents α, β, γ set to 1.

[6] Note that the blow up maps the origin $0 \in \mathbb{R}^{N+1}$ to the sphere $\mathbb{S}^N \times \{0\}$ while the blow down does the opposite, hence the names.

Definition 9 *Consider $\mathcal{X}$ as in* (7) *and Φ as in* (8). *The* blow up *of $\mathcal{X}$ is a vector field $\bar{\mathcal{X}} : \mathcal{B} \to T\mathcal{B}$ induced by Φ in the sense $\bar{\mathcal{X}} = D\Phi^{-1} \circ \mathcal{X} \circ \Phi$, where $D\Phi$ denotes the differential of Φ.*

It may happen that the vector field $\bar{\mathcal{X}}$ degenerates along $\mathcal{Z}$. In such a case we define the *desingularized vector field* $\tilde{\mathcal{X}}$ as

$$\tilde{\mathcal{X}} = \frac{1}{\bar{r}^m}\bar{\mathcal{X}},$$

for some well suited $m \in \mathbb{N}$ so that $\tilde{\mathcal{X}}$ is not degenerate, and is well defined along $\mathcal{Z}$. Note that the vector fields $\bar{\mathcal{X}}$ and $\tilde{\mathcal{X}}$ are equivalent on $\mathbb{S}^N \times \{\bar{r} > 0\}$. Moreover, if the weights (α, β, γ) are well chosen, the singularities of $\tilde{\mathcal{X}}|_{\bar{r}=0}$ are partially hyperbolic or even hyperbolic, making the analysis of $\tilde{\mathcal{X}}$ simpler than that of $\mathcal{X}$. Due to the equivalence between $\mathcal{X}$ and $\tilde{\mathcal{X}}$, one obtains all the local information of $\mathcal{X}$ around $0 \in \mathbb{R}^{N+1}$ from the analysis of $\tilde{\mathcal{X}}$ around $\mathcal{Z}$.

While doing computations, it is more convenient to study the vector field $\tilde{\mathcal{X}}$ in charts. A chart is a parametrization of a hemisphere of $\mathcal{B}$ and is obtained by setting one of the coordinates $(\bar{x}, \bar{z}, \bar{\varepsilon}) \in \mathbb{S}^N$ to ± 1 in the definition of Φ. In this article, we only use the chart $\kappa_\varepsilon = \{\bar{\varepsilon} = 1\}$ as it is in such chart where the singular behavior of the system is overcome, see Proposition 10. However, we remark that the analysis in the other charts may prove useful to design better controllers, or even necessary for the problem of path following and trajectory tracking of SFCSs with non-hyperbolic points.

5 Geometric desingularization of a slow-fast control system

In this section we perform the geometric desingularization of (5). Without loss of generality we can write $f(x, z, \varepsilon) = A + L(x, z) + F(x, z, \varepsilon)$, where $A = f(0, 0, 0)$, $L(x, z)$ is a linear map, i.e., $L(x, z) = L_1 x + L_2 z$ with $L_1 \in \mathbb{R}^{(n_s)\times(n_s)}$, $L_2 \in \mathbb{R}^{n_s}$, and $F(x, z, \varepsilon)$ stands for all the higher order terms and satisfies, due to Assumption **A4**, $F(0, 0, \varepsilon) = 0$. Thus, (5) is rewritten as

$$\mathcal{X} : \begin{cases} x' &= \varepsilon\left(A + L(x, z) + Bu(x, z, \varepsilon) + F\right) \\ z' &= -G_k(x, z) + H(x, z, \varepsilon), \\ \varepsilon' &= 0, \end{cases} \tag{9}$$

where for shortness of notation we omit the dependence of B and F on (x, z, ε). The blow up map is defined by

$$x = \bar{r}^\alpha \bar{x},\ z = \bar{r}^\gamma \bar{z},\ \varepsilon = \bar{r}^\rho \bar{\varepsilon},$$

where $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_{n_s})$, $\bar{z} \in \mathbb{R}$, $\bar{\varepsilon} \geq 0$, $\sum_{i=1}^{n_s} \bar{x}_i^2 + \bar{z}^2 + \bar{\varepsilon}^2 = 1$, $\alpha \in \mathbb{Z}^{n_s}$, $(\gamma, \rho) \in \mathbb{Z}^2$ and $\bar{r} \in [0, r_0)$. In principle, we could set all the exponents (α, γ, ρ) to 1, but this would require more than one coordinate transformation to completely desingularize (9) [2, 24]. Noting that $G_k(x, z)$ is quasi-homogeneous [1, 13] of type $(k, k-1, \ldots, 1)$, we choose a quasi-homogeneous blow up map defined by

$$x_i = \bar{r}^{k-i+1}\bar{x}_i,\ x_j = \bar{r}^k \bar{x}_j,\ z = \bar{r}\bar{z},\ \varepsilon = \bar{r}^{2k-1}\bar{\varepsilon},$$

for $i = 1, \ldots, k-1$ and $j = k, \ldots, n_s$. We remark that the change of coordinates for (x_i, z) $(i = 1, \ldots, k-1)$ follows from the quasihomogeneity type of G_k. On the other hand, the change of coordinates of x_j $(j = k, \ldots, n_s)$ could be arbitrary, after all the coordinates $(x_k, \ldots, x_{n_s})$ have no influence in the nature of the non-hyperbolic origin. It just turns out that the proposed one is convenient.

For shortness of notation let $\alpha = (k, \ldots, 2, k, \ldots, k)$ so that $x = \bar{r}^\alpha \bar{x}$.

Proposition 10 *The desingularized vector field $\tilde{\mathcal{X}}^{\bar{\varepsilon}}$, which corresponds to the blow up of* (9) *in the chart $\kappa_{\bar{\varepsilon}}$, can be written as*

$$\tilde{\mathcal{X}}^{\bar{\varepsilon}} : \begin{cases} \bar{r}' &= 0 \\ \bar{x}_i' &= \bar{r}^{i-1}(A_i + (\bar{B}\bar{u})_i + O(\bar{r})) \\ \bar{x}_j' &= A_j + (\bar{B}\bar{u})_j + O(\bar{r}) \\ \bar{z}' &= -G_k(\bar{x}, \bar{z}) + O(\bar{r}), \end{cases} \tag{10}$$

for $i = 1, \ldots, k-1$, $j = k, \ldots, n_s$, and where $\bar{B} = \bar{B}(\bar{r}, \bar{x}, \bar{z}) = B(\bar{r}^\alpha \bar{x}, \bar{r}\bar{z}, \bar{r}^{2k-1})$, similarly for $\bar{u}$, and where the subscript denotes the component of the corresponding vector.

PROOF. From $\varepsilon = \bar{r}^{2k-1}$, it follows that $\bar{r}' = 0$. Next, note that when applying the change of coordinates we have

$$L(\bar{r}^\alpha \bar{x}, \bar{r}\bar{z}) = \bar{r} L(\bar{r}^{\alpha-1}\bar{x}, \bar{z}) \in O(\bar{r}),$$

where $\bar{r}^{\alpha-1}\bar{x} = (\bar{r}^{k-1}\bar{x}_1, \ldots, \bar{r}\bar{x}_{k-1})$. Also,

$$Bu = \underbrace{B(\bar{r}^\alpha \bar{x}, \bar{r}\bar{z}, \bar{r}^{2k-1})}_{\bar{B}(\bar{r}, \bar{x}, \bar{z})}\underbrace{u(\bar{r}^\alpha \bar{x}, \bar{r}\bar{z}, \bar{r}^{2k-1})}_{\bar{u}(\bar{r}, \bar{x}, \bar{z})}$$

and

$$F = F(\bar{r}^\alpha \bar{x}, \bar{r}\bar{z}, \bar{r}^{2k-1}) \in O(\bar{r}^2).$$

Recall that F is at least quadratic, that is why the above order. Then, taking into account the above expressions, we get

$$\begin{aligned} x_i' = \bar{r}^{k-i+1}\bar{x}_i' &= \bar{r}^{2k-1}(A_i + (\bar{B}\bar{u})_i + O(\bar{r})) \\ \bar{x}_i' &= \bar{r}^{k+i-2}(A_i + (\bar{B}\bar{u})_i + O(\bar{r})), \\ x_j' = \bar{r}^k \bar{x}_j' &= \bar{r}^{2k-1}(A_j + (\bar{B}\bar{u})_j + O(\bar{r})) \\ \bar{x}_j' &= \bar{r}^{k-1}(A_j + (\bar{B}\bar{u})_j + O(\bar{r})), \end{aligned} \tag{11}$$

for $i = 1, \dots, k-1$, $j = k, \dots, n_s$, and where the subscript denotes the element of the corresponding vector. On the other hand

$$\begin{aligned} z' = \bar{r}\bar{z}' &= -G_k(\bar{r}^\alpha \bar{x}, \bar{r}\bar{z}) + H(\bar{r}^\alpha \bar{x}, \bar{r}\bar{z}, \bar{r}^{2k-1}) \\ &= -\bar{r}^k G_k(\bar{x}, \bar{z}) + O(\bar{r}^{k+1}) \\ \bar{z}' &= \bar{r}^{k-1}(-G_k(\bar{x}, \bar{z}) + O(\bar{r})). \end{aligned} \tag{12}$$

Here we remark that $G_k(\bar{r}^\alpha \bar{x}, \bar{r}\bar{z}) = \bar{r}^k G_k(\bar{x}, \bar{z})$ due to quasihomogeneity and $H \in O(\bar{r}^{k+1})$ because it contains the higher order terms. Finally, we divide the right hand sides of (11) and (12) by $\bar{r}^{k-1}$ to obtain

$$\begin{aligned} \bar{r}' &= 0 \\ \bar{x}_i' &= \bar{r}^{i-1}(A_i + (\bar{B}\bar{u})_i + O(\bar{r})) \\ \bar{x}_j' &= A_j + (\bar{B}\bar{u})_j + O(\bar{r})) \\ \bar{z}' &= -G_k(\bar{x}, \bar{z}) + O(\bar{r}), \end{aligned}$$

as stated. □

The stability properties of the blown up vector field $\tilde{\mathcal{X}}^{\bar{\varepsilon}}$ of (10) are carried over similar properties into the original SFS. The main argument is that "stability should be invariant under changes of coordinates", as is analyzed in the next section.

6 Controller design via geometric desingularization

In this section we show how to use geometric desingularization to stabilize a non-hyperbolic point of a SFCS. The method is to design a controller in the blown up space and then to blow such controller down. We do this in the central chart $\kappa_{\bar{\varepsilon}}$ as argued above.

Remark 11 *Note that in* (10) $\bar{r}$ *is a regular perturbation parameter. Thus, the idea is to solve first the stabilization problem for* $\bar{r} = 0$ *and then use regular perturbation arguments [26] to guarantee the stability of the origin of* (10) *for* $\bar{r} \geq 0$ *sufficiently small.*

The main argument to relate the stability of the blown up vector field $\tilde{\mathcal{X}}^{\bar{\varepsilon}}$ and $\mathcal{X}$ is the following.

Proposition 12 *Consider a SFCS given by* (9) *and its blow up version given by* (10). *If for each* $\bar{r} \in (0, \bar{r}_0]$ *there exists a controller* $\bar{u}$ *that renders the point* $(\bar{x}, \bar{z}) = (0, 0)$ *stable (resp. locally a.s.* [7] *, resp. globally a.s.) for* $\tilde{\mathcal{X}}^{\bar{\varepsilon}}$*, then for each* $\varepsilon \in (0, \bar{r}^{1/(2k-1)}]$ *there exists a controller* u *that renders the point* $(x, z) = (0, 0)$ *stable (resp. locally a.s., resp. globally a.s.) for* $\mathcal{X}$.

[7] asymptotically stable

PROOF. Although our concern is for local stability, we shall prove the statement for global asymptotic stability, as it is the most interesting one. Naturally, the proof for the other cases follows a similar line of thought. Let $\bar{r} = \bar{r}_0 > 0$ be fixed. Then the blow up change of coordinates be defined by

$$\phi(\bar{x}, \bar{z}) = (\bar{r}_0^\alpha \bar{x}, \bar{r}_0 \bar{z}) = (x, z). \tag{13}$$

Note that ϕ is a diffeomorphism with positive definite Jacobian. Next, let $\bar{u} = \bar{u}(\bar{x}, \bar{z})$ be chosen such that it renders $(\bar{x}, \bar{z}) = (0, 0)$ globally asymptotically stable for the dynamics of $\tilde{\mathcal{X}}^{\bar{\varepsilon}}$, and define u as the blow down of $\bar{u}$, that is $u(x, z) = \phi \circ \bar{u}(\bar{x}, \bar{z})$. So, we have that the closed-loop systems $\tilde{\mathcal{X}}^{\bar{\varepsilon}}$ and $\mathcal{X}$ are related by $\tilde{\mathcal{X}}^{\bar{\varepsilon}}(\bar{x}, \bar{z}) = \frac{1}{\bar{r}^m} D\phi^{-1} \circ \mathcal{X} \circ \phi(\bar{x}, \bar{z})$.

On the other hand, the hypothesis that $(\bar{x}, \bar{z}) = (0, 0)$ is G.A.S. for $\tilde{\mathcal{X}}^{\bar{\varepsilon}}$ implies that there exists a $\bar{r}$-family of Lyapunov functions $\bar{V}_{\bar{r}}(\bar{x}, \bar{z})$ satisfying

- $\bar{V}_{\bar{r}} > 0$ for all $(\bar{x}, \bar{z}) \in \mathbb{R}^{n_s+1} \setminus \{0\}$,
- $\bar{V}_{\bar{r}}' < 0$ for all $(\bar{x}, \bar{z}) \in \mathbb{R}^{n_s+1} \setminus \{0\}$,
- $\bar{V}_{\bar{r}}$ is radially unbounded.

Define the ε-family of Lyapunov candidate functions $V_\varepsilon = \bar{V}_{\bar{r}} \circ \phi^{-1}$, where $\varepsilon = \bar{r}^{2k-1}$. From the properties of ϕ (13), namely that ϕ is a diffeomorphism with positive definite Jacobian, it follows that $V_\varepsilon > 0$ and $V_\varepsilon' < 0$ for all $(x, z) \in \mathbb{R}^{n_s+1} \setminus \{0\}$. Therefore, V_ε is an ε-family of Lyapunov functions. Finally, let $||(\bar{x}, \bar{z})|| \to \infty$, which clearly implies that $||(x, z)|| = ||\phi(\bar{x}, \bar{z})|| \to \infty$. From the definition of V_ε we have $V_\varepsilon(x, z) = \bar{V}_{\bar{r}} \circ \phi^{-1}(x, z) = \bar{V}_{\bar{r}}(\bar{r}^\alpha \bar{x}, \bar{r}\bar{z})$, $\bar{r} > 0$, which in fact shows that V_ε is also radially unbounded. □

Proposition 12 means that we can design controllers to stabilize a SFCS by designing them in the blown up space. The way the controller $\bar{u}$ is actually designed depends on the specific context of the problem. Below we present a particularly interesting case where even though the origin is non-hyperbolic and the fast variable is not actuated, we are able to inject a hyperbolic behavior by actuating only the slow variables.

6.1 Hyperbolicity injection

In this section we design a controller which induces a hyperbolic behavior in (10) around the origin. We do this via the backstepping algorithm [32].

Proposition 13 *Consider* (10). *If* $\bar{u}$ *is designed such* *that*

$$(\bar{B}\bar{u})_1(\bar{r},\bar{x},\bar{z}) = -A_1 + \left(\frac{\partial G_k}{\partial \bar{z}} - c_0 - c_1\right) G_k + (1+c_0c_1)\bar{z} + \sum_{i=2}^{k-1} c_i\bar{x}_i\bar{z}^{i-1} \tag{14}$$
$$(\bar{B}\bar{u})_i(\bar{r},\bar{x},\bar{z}) = -A_i - \bar{r}^{1-i}c_i\bar{x}_i$$
$$(\bar{B}\bar{u})_j(\bar{r},\bar{x},\bar{z}) = -A_j - c_j\bar{x}_j$$

for $i = 2,\dots,k-1$, $j = k,\dots,n_s$ *and where all the* $c_\bullet$*'s are positive constants, then the origin of* (10) *is rendered locally asymptotically stable for* $\bar{r} \geq 0$ *sufficiently small.*

PROOF. Along this proof, unless otherwise stated, $i = 1,\dots,k-1$ and $j = k,\dots,n_s$.

First, let $(\bar{B}\bar{u})_i = -A_i + \bar{r}^{1-i}\bar{v}_i(\bar{x},\bar{z})$ and $(\bar{B}\bar{u})_j = -A_j + \bar{v}_j(\bar{x},\bar{z})$. Substituting this expression into (10), and restricting to $\{\bar{r}=0\}$ we get

$$\begin{aligned} \bar{x}_i' &= \bar{v}_i(\bar{x},\bar{z}) \\ \bar{x}_j' &= \bar{v}_j(\bar{x},\bar{z}) \\ \bar{z}' &= -G_k(\bar{x},\bar{z}). \end{aligned} \tag{15}$$

Note that we can choose $\bar{v}_i$ and $\bar{v}_j$ such that the dynamics of $(\bar{x}_i,\bar{z})$ are uncoupled from those of $\bar{x}_j$ in (15). We do this by setting $v_j(\bar{x},\bar{z}) = -c_j\bar{x}_j$ with $c_j > 0$. So, the rest of the proof concerns the stabilization of $(\bar{x}_i,\bar{z})$.

Next, we consider the fast equation $\bar{z}' = -G_k(\bar{x},\bar{z})$, and treat $\bar{x}_1$ as a virtual controller. So let $\bar{x}_1 = \alpha(\bar{x},\bar{z}) = -(\bar{z}^k + \sum_{i=2}^{k-1}\bar{x}_i\bar{z}^{i-1}) + c_0\bar{z}$. Then, the closed-loop fast equation reads as $\bar{z}' = -c_0\bar{z}$, for which the origin is clearly exponentially stable. Moreover, making the relation with the original (slow-fast) coordinates, we see that $\bar{z} = 0$ (equivalent to $z = 0$) is a "normally hyperbolic set" of the aforementioned (virtual) closed-loop system. This is the reason why we call this design "hyperbolicity injection".

Next, let $\zeta = \bar{x}_1 - \alpha$, and $Y = (Y_1,\dots,Y_{k-1}) = (\zeta,\bar{x}_2,\dots,\bar{x}_{k-1})$. Then (15) is rewritten as

$$\begin{aligned} Y_1' &= \bar{v}_1 + \sum_{i=2}^{k-1}\bar{z}^{i-1}\bar{v}_i - \underbrace{\left(\frac{\partial G_k}{\partial \bar{z}} - c_0\right)(Y_1 + c_0\bar{z})}_{W(\bar{x},\bar{z})} \\ Y_i' &= \bar{v}_i \\ \bar{z}' &= -(Y_1 + c_0\bar{z}), \end{aligned} \tag{16}$$

where $i = 2,\dots,k-1$, and although we are changing coordinates, we recycle the notation to avoid introducing new functions (later we come back to the original coordinates). Now we design $\bar{v}_i(\bar{x},\bar{z})$ for $i = 1,\dots,k-1$ to render the origin of (16) asymptotically stable. For this, let us use a Lyapunov candidate function of the form

$$V = \frac{1}{2}\bar{z}^2 + \frac{1}{2}Y^TY.$$

Then

$$\begin{aligned} V' &= \bar{z}\bar{z}' + \sum_{i=1}^{k-1} Y_iY_i' \\ &= -c_0\bar{z}^2 + Y_1\left(-\bar{z} - W + \sum_{i=2}^{k-1}\bar{z}^{i-1}\bar{v}_i + \bar{v}_1\right) \\ &\quad + \sum_{i=2}^{k-1} Y_i\bar{v}_i. \end{aligned}$$

With the above expression, let us choose

$$\begin{aligned} \bar{v}_1 &= W + \bar{z} + \sum_{i=2}^{k-1} c_iY_i\bar{z}^{i-1} - c_1Y_1 \\ \bar{v}_i &= -c_iY_i, \end{aligned}$$

for $i = 2,\dots,k-1$, guaranteeing that V is a Lyapunov function with $V(\bar{x},\bar{z}) > 0$ and $V'(\bar{x},\bar{z}) < 0$, and we can choose constants $c_i > 0$ to tune the convergence rate of $(Y,\bar{z}) \to (0,0)$. Note that $(Y,\bar{z}) \to 0$ implies (besides $(\bar{x}_2,\dots,\bar{x}_{k-1},\bar{z}) \to 0 \in \mathbb{R}^{k-1}$) that $\bar{x}_1 \to \alpha$. However, it is clear that $(\bar{x}_2,\dots,\bar{x}_{k-1},\bar{z}) \to 0$ implies $\alpha \to 0$. Moreover, the closed-loop system turns out to be linear. Thus overall we accomplish exponential stability of the origin of (16).

Remark 14 *The constant* c_1 *dictates the rate at which* $Y_1 = \zeta = \bar{x}_1 - \alpha \to 0$*. Thus in order to ensure that this control algorithm works, we must choose* c_1 *sufficiently larger than all the other* c_i *constants, that is* $c_i \ll c_1$ *for all* $i = 0,2,\dots,k-1$.

Finally, returning to $(\bar{x},\bar{z})$ coordinates and joining our previous analysis we get

$$(\bar{B}\bar{u})_1(\bar{r},\bar{x},\bar{z}) = -A_1 + \left(\frac{\partial G_k}{\partial \bar{z}} - c_0 - c_1\right) G_k + (1+c_0c_1)\bar{z} + \sum_{i=2}^{k-1} c_i\bar{x}_i\bar{z}^{i-1}$$
$$(\bar{B}\bar{u})_i(\bar{r},\bar{x},\bar{z}) = -A_i - \bar{r}^{1-i}c_i\bar{x}_i$$
$$(\bar{B}\bar{u})_j(\bar{r},\bar{x},\bar{z}) = -A_j - c_j\bar{x}_j$$

for $i = 2,\dots,k-1$ and $j = k,\dots,n_s$ as stated. The asymptotic stability result for $\bar{r} > 0$ sufficiently small follows from regular perturbation theory [26] and the fact that the origin of the closed-loop system is a hyperbolic equilibrium point. □

6.2 Proof of Theorem 6

The proof follows from Proposition 13, the corresponding blow down of the controller $\bar{u}$, that is $u(x,z,\varepsilon) = \Phi \circ \bar{u} \circ \Phi^{-1}(x,z,\varepsilon)$, and Proposition 12. That is, to obtain (6) we apply the coordinate transformation

$$\bar{r} = \varepsilon^{\frac{1}{2k-1}},\ \bar{x}_i = \varepsilon^{\frac{-k+i-1}{2k-1}} x_i,\ \bar{x}_j = \varepsilon^{\frac{-k}{2k-1}} x_j,\ \bar{z} = \varepsilon^{\frac{-1}{2k-1}} z,$$

for $i = 1, \dots, k-1$ and $j = k, \dots, n_s$ to (14). □

7 Numerical simulations

7.1 Didactic example

To showcase the controller we designed in a highly degenerate scenario, let us consider the slow-fast control problem

$$\begin{aligned} \dot{x} &= \mathbb{1} + u(x,z,\varepsilon) \\ \varepsilon z &= -(z^4 + x_3 z^2 + x_2 z + x_1), \end{aligned} \tag{17}$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $u \in \mathbb{R}^3$, $z \in \mathbb{R}$, and $\mathbb{1} = [1\ 1\ 1]^T$. Note that (17) is of the form of (5) with $k = 4$, $n_s = 3$ and B the identity matrix. It is readily seen that the origin is unstable for the open-loop dynamics. So, using Theorem 6 we shall design a controller that locally stabilizes the origin. According to (6), the appropriate controller is

$$\begin{aligned} u_1 &= -1 + \varepsilon^{-1/7}(1 + c_0 c_1) z + \varepsilon^{-4/7}(c_2 x_2 z + c_3 x_3 z^2) \\ &\quad \varepsilon^{-1} \left(\frac{\partial G_4}{\partial z} - \varepsilon^{3/7}(c_0 + c_1) \right) G_4 \\ u_2 &= -1 - c_2 \varepsilon^{-4/7} x_2 \\ u_3 &= -1 - c_3 \varepsilon^{-4/7} x_3. \end{aligned} \tag{18}$$

Figure 1, shows the closed-loop response using (18). We note that the large spike near $t = 0$ is due to the "hyperbolicity injection". Recall that the first step in the design of the controller $\bar{u}$ (in the blow up space) is to assume $\bar{x}_1 = -(\bar{z}^4 + \bar{x}_3 \bar{z}^2 + \bar{x}_2 \bar{z}) + c_0 \bar{z}$. In order to achieve this the gain c_1 is chosen much larger than the other constants to let $\bar{x}_1 \to -(\bar{z}^4 + \bar{x}_3 \bar{z}^2 + \bar{x}_2 \bar{z}) + c_0 \bar{z}$ fast. Thus, the spike is precisely this mapping. Afterwards the trajectories reach the equilibrium point in an exponential fashion since the resulting vector field is linear, see Proposition 13 and its proof for full details.

7.2 Stabilization at a fold point of a nonlinear electric circuit

Let us consider the electric circuit Σ_1 as shown in Figure 2.(a), where the Capacitor C and Inductor L are usual

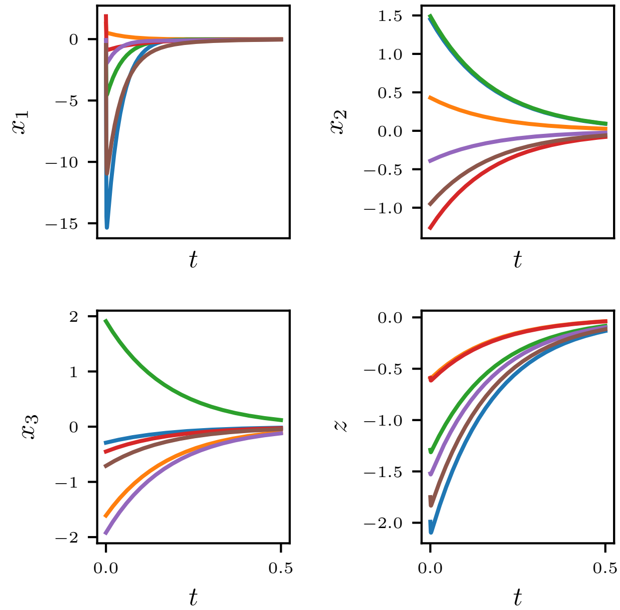

Fig. 1. Simulation of the closed loop system (17) under the controller of Theorem 6. For this simulation we have used gains $c_0 = c_2 = c_3 = 1$, $c_1 = 300$ and $\varepsilon = 0.05$; and six random initial conditions within $(-2, 2)$ for each state.

elements, but the Resistor R is assumed to be nonlinear, that is $I_R = f(V_R)$ where f is some nonlinear function and I_R and V_R denote the current and voltage of R respectively, see Example 5 of [34].

For the purpose of this example we shall assume that

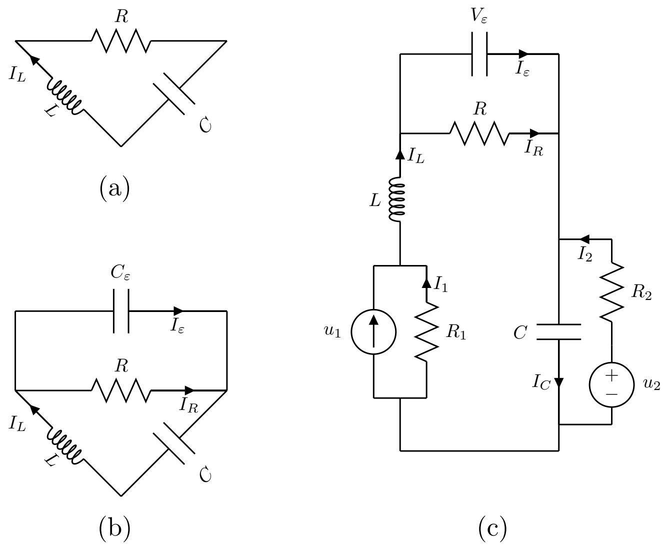

Fig. 2. (a) Electric circuit Σ_1 with a nonlinear resistive load. (b) The regularization of circuit (a), denoted by Σ_2. (c) The controlled circuit Σ_3.

$f(V_R) = \frac{1}{3}V_R^3 - V_R$, and then the characteristic curve of the nonlinear Resistor R is as depicted in Figure 3. We remark that the chosen type of nonlinear behavior is qualitatively the same as the one in [34], and also appears in several other nonlinear elements, such as tunnel diodes [28].

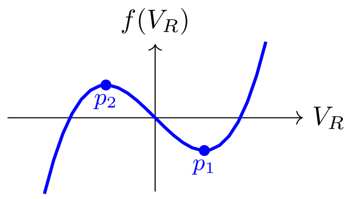

Fig. 3. The characteristic curve of the nonlinear Resistor R.

The characteristic curve depicted in Figure 3 has two fold points $p_{1,2}$ located at $(V_R, I_R) = (\pm 1, \mp\frac{2}{3})$. At these points the differential equation describing the behavior of the circuit becomes singular. To overcome this it is proposed to regularize Σ_1 by adding a parasitic capacitance in parallel to R as depicted in Figure 2.(b) (electric circuit Σ_2), see Example 6 of [34] for more details, and [12]. The capacitance C_ε is assumed to be small, e.g. $C_\varepsilon = \varepsilon$. Thus, as $\varepsilon \to 0$, the behavior of of circuit Σ_2 approaches that of Σ_1. Let $(x_1, x_2, z) = (I_L, V_C, V_\varepsilon)$, where V_ε denotes the voltage at the capacitor C_ε. The equations describing the behavior of the circuit Σ_2 read as

$$\begin{aligned} L\dot{x}_1 &= -z - x_2 \\ C\dot{x}_2 &= x_1 \\ \varepsilon\dot{z} &= -\frac{1}{3}z^3 + z + x_1. \end{aligned} \tag{19}$$

We immediately note that the corresponding critical manifold is actually given by the characteristic equation of the nonlinear resistor, namely

$$S = \left\{ (x_1, x_2, z) \in \mathbb{R}^3 \,|\, x_1 = \frac{1}{3}z^3 - z \right\}.$$

Moreover, it is straightforward to see that the region of S between p_1 and p_2 is repelling while the rest of S is attracting. Furthermore, one can easily show that (19) has three equilibrium points $\{q_1, q_2, q_3\} = \left\{(0,0,0), (0, -\sqrt{3}, \sqrt{3}), (0, \sqrt{3}, -\sqrt{3})\right\}$, where q_1 is unstable, while q_2 and q_3 are stable. With this information we can qualitatively describe the dynamics of (19) as follows: for initial conditions away from S, the trajectories of (19) quickly approach a stable region of S, and then evolve along it. Here two things may happen, trajectories may converge to an equilibrium point q_2 or q_3, or they can approach a fold point. When a trajectory reaches a fold point, then the trajectory jumps towards a stable region of S and then follows the same behavior as as described before. A sample of trajectories of (19) is provided in Figure 4.

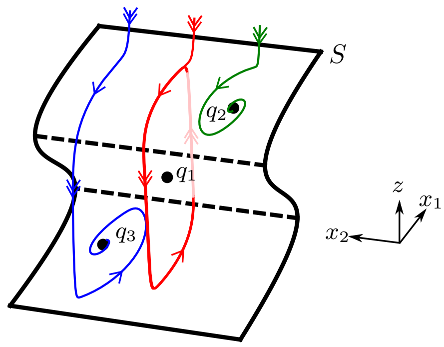

Fig. 4. Trajectories of the open-loop system (19). The surface is the critical manifold S while dashed lines represent lines of folds, and the dots stand for the equilibrium points. Note the existence of a limit cycle, which can be conjectured from the stability of q_2 and q_3. However such analysis falls off the scope of this document and shall not be discussed further.

Our goal, however, is to set as operating point a fold point. In order to do this we introduce controllers as depicted in Figure 2.(c) leading to circuit Σ_3

Let us choose, for example, the operating point $P = (x_1, x_2, z) = (-\frac{2}{3}, 0, 1)$. Then let us define new coordinates $(X_1, X_2, Z) = (-x_1 - \frac{2}{3}, x_2, z - 1)$, with which the behavior of circuit Σ_3 is described by

$$\begin{aligned} L\dot{X}_1 &= 1 - \frac{2}{3}R_1 - R_1X_1 + X_2 + Z - R_1u_1 \\ C\dot{X}_2 &= -\frac{2}{3} - X_1 + \frac{X_2}{R_2} - \frac{u_2}{R_2} \\ \varepsilon\dot{Z} &= -(Z^2 + X_1) - \frac{1}{3}Z^3. \end{aligned}$$

According to Theorem 6 the controller that stabilizes P is given by

$$\begin{aligned} u_1 &= -\frac{L}{R_1}\left(-\frac{1}{L} + \frac{2}{3L}R_1 - \frac{c_1}{\varepsilon^{2/3}}X_1 + \frac{c_0c_1 + 1}{\varepsilon^{1/3}}Z + \beta \right) \\ u_2 &= -CR_2\left(\frac{2}{3C} - \frac{c_1}{\varepsilon^{2/3}}X_2 \right), \end{aligned}$$

where $\beta = \varepsilon^{-2/3}\left(-c_1Z^2 + (2\varepsilon^{-1/3}Z - c_0)(Z^2 + X_1)\right)$.

To witness the effects of the controller, let us choose parameters: $C = 1$F, $L = 1$H, $R_1 = R_2 = 1\Omega$, $\varepsilon = 0.05$, and controller gains c_0 and $c_1 = 10$. Next we choose initial conditions near the limit cycle, to allow the system oscillate. For simulation purposes we let the system evolve 10 seconds in open-loop. Then, at $t = 10$s we activate the controller. The results are shown in Figure 5, where we see that when the controller takes action, the trajectories quickly converge to the operating point P.

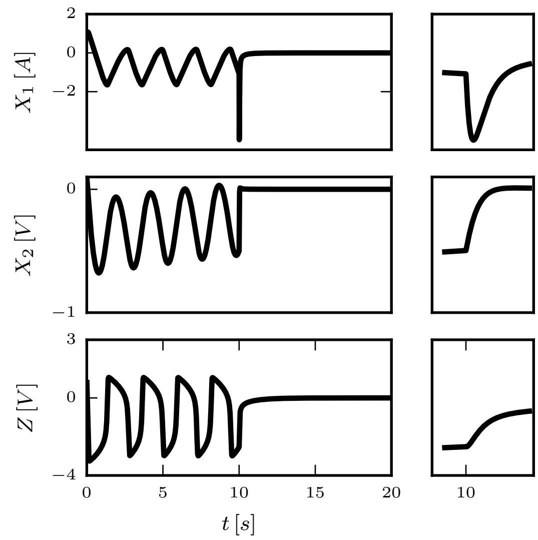

Fig. 5. Left: Simulation results of the controlled circuit Σ_3 depicted in Figure 4.(c) and with the controller of Theorem 6. Right: Zoom-in for the interval $t \in [9.975, 10.075]$. The quick jump is produced by the fact that the controller forces $x \to -z^2+\varepsilon^{1/3}c_0 z$ (see the details in the proof of Proposition 13 where we argue that $\bar{x} \to \alpha(\bar{z})$ fast enough).

8 Conclusions

In this paper we have introduced the geometric desingularization technique to control systems. The main contribution is a controller design method to stabilize non-hyperbolic points of a class of slow-fast systems with one fast direction. The main feature of our controller is that it is able to cope with singularities of arbitrary degeneracy. Another essential characteristic of our contribution is that the controller only actuates on the slow variables, making it more suitable for certain applications. As a case study, we have provided a controller based on the backstepping algorithm that renders the origin of slow-fast control systems locally asymptotically stable.

Further research directions in view of the potential applications include: the assumption that the slow system is under-actuated, output feedback control, trajectory and path-following along sets with non-hyperbolic points such as canards and mixed-mode oscillations. Besides, the relation between the choice of blow up and the performance of the controllers deserves further investigation. Another challenging framework is to consider model-order reduction techniques, stabilization, consensus, etc. for networks of SFSs with non-hyperbolic points [16].

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